S^3EPB

SLAC Summer Seminar
on Electron and Photon Beams

Introduction to electron and photon beam physics

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SLAC and Stanford University

August 03, 2015

Lecture Plan

- **Electron beams (1.5 hrs)**

- **Photon or radiation beams (1 hr)**

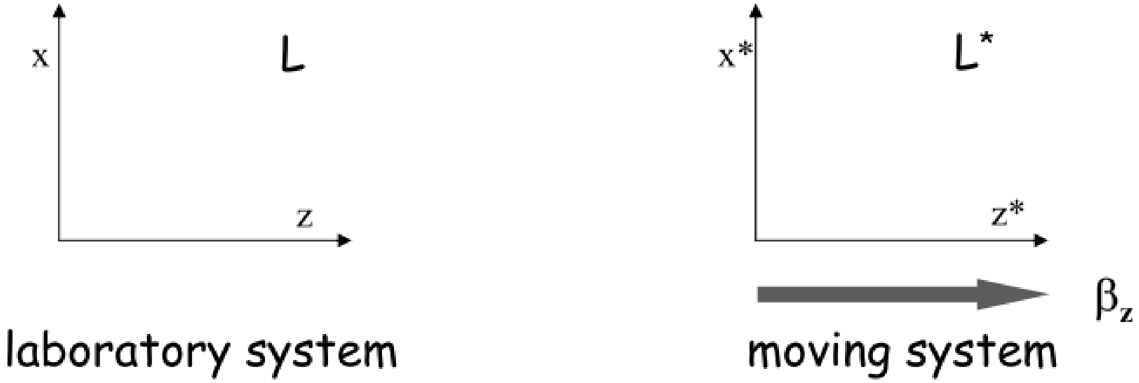
References:

1. J. D. Jackson, Classical Electrodynamics (Wiley, New York, third edition, 1999).
2. Helmut Wiedemann, Particle Accelerator Physics (Springer-Verlag, 2003).
3. *Andrew Sessler and Edmund Wilson,* Engine of Discovery (World Scientific, 2007).
4. *David Attwood, Soft X-rays and Extreme Ultraviolet Radiation (Cambridge, 1999)*
5. *Peter Schmüser, Martin Dohlus, Jörg Rossbach,* Ultraviolet and Soft X-Ray Free-Electron Lasers (Springer-Verlag, 2008).
6. Kwang-Je Kim, Zhirong Huang, Ryan Lindberg, Synchrotron Radiation and Free-Electron Lasers for Bright X-ray Sources, USPAS lecture notes 2013.
7. Gennady Stupakov, Classical Mechanics and Electromagnetism in Accelerator Physics, USPAS Lecture notes 2011.
8. Images from various sources and web sites.

Electron beams

- Primer on special relativity and E&M
- Accelerating electrons
- Transporting electrons
- Beam emittance and optics
- Beam distribution function

Lorentz Transformation



$$\begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & \beta\gamma \\ 0 & 0 & \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x^* \\ y^* \\ z^* \\ ct^* \end{pmatrix}$$

$$\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \sqrt{1 - \frac{1}{\gamma^2}}$$

$$\gamma = \frac{E_e}{mc^2} = \frac{E_e[\mathrm{GeV}]}{0.511 \times 10^{-3}} = 1957\, E_e[\mathrm{GeV}]$$

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} \approx 1 - \frac{1}{2\gamma^2},$$

$$1 - \beta \approx 5 \times 10^{-8} \quad \text{for} \quad E_e = 1.5 \text{ GeV}$$

Length Contraction and Time Dilation

- Length contraction: an object of length Δz^* aligned in the moving system with the z^* axis will have the length Δz in the lab frame

$$\boxed{\Delta z = \frac{\Delta z^*}{\gamma}}$$

- Time dilation: Two events occurring in the moving system at the same point and separated by the time interval Δt^* will be measured by the lab observers as separated by Δt

$$\boxed{\Delta t = \gamma \Delta t^*}$$

Energy, Mass, Momentum

- Energy

$$\boxed{E = T + mc^2}$$

Kinetic energy **Rest mass energy**

- Electrons rest mass energy 511 keV (938 MeV for protons), 1eV = 1.6 × 10^{-19} Joule

- Momentum

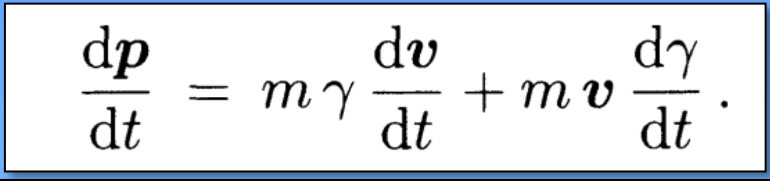
$$\boxed{\boldsymbol{p} = \gamma \boldsymbol{\beta} mc}$$

- Energy and momentum

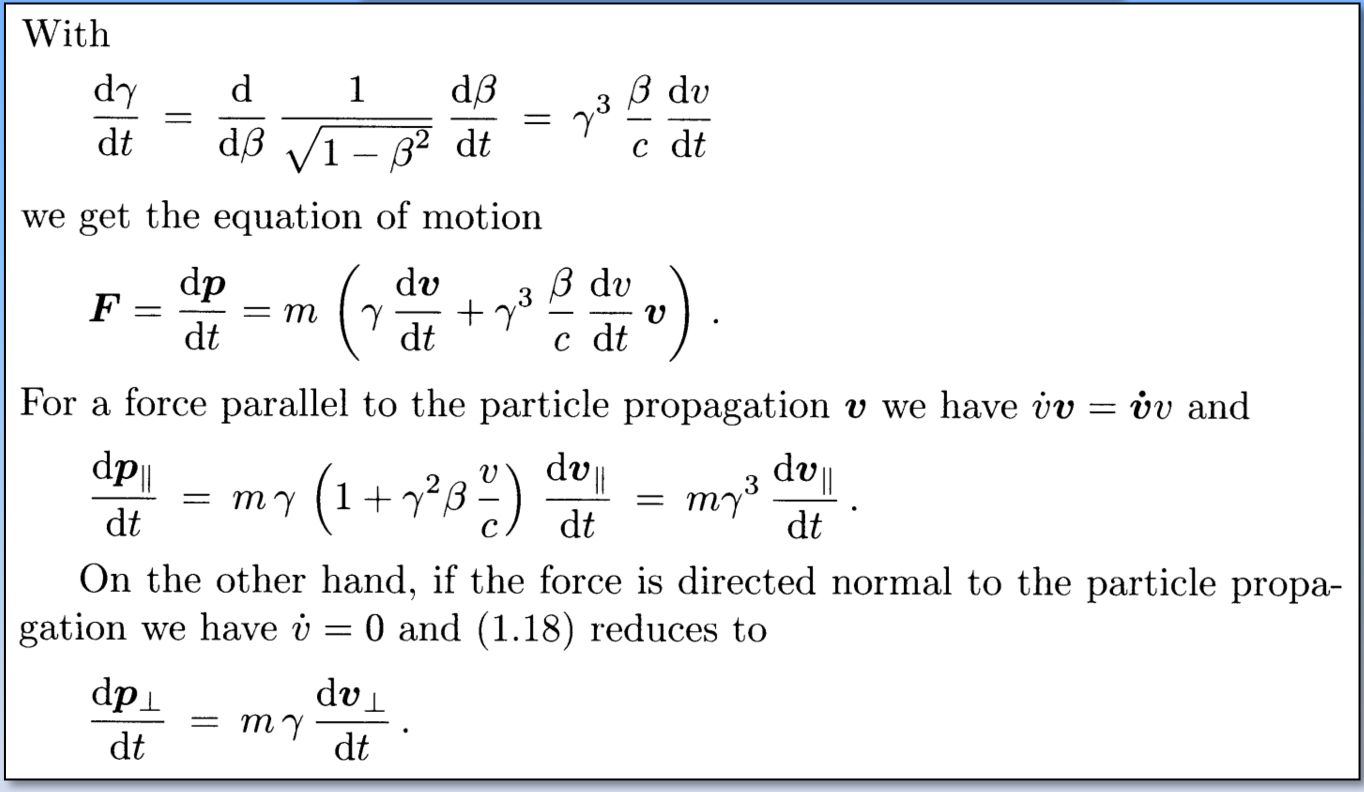
$$\boxed{\begin{array}{l} E^2 = p^2c^2 + m^2c^4, \\ E = \gamma mc^2. \end{array}}$$

Relativistic acceleration

- Momentum change



$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = m\,\gamma\,\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + m\,\boldsymbol{v}\,\frac{\mathrm{d}\gamma}{\mathrm{d}t}\,.$$



With

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}\beta}\frac{1}{\sqrt{1-\beta^2}}\frac{\mathrm{d}\beta}{\mathrm{d}t} = \gamma^3\,\frac{\beta}{c}\,\frac{\mathrm{d}v}{\mathrm{d}t}$$

we get the equation of motion

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = m\left(\gamma\,\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \gamma^3\,\frac{\beta}{c}\,\frac{\mathrm{d}v}{\mathrm{d}t}\,\boldsymbol{v}\right).$$

For a force parallel to the particle propagation $\boldsymbol{v}$ we have $\dot{v}\boldsymbol{v} = \dot{\boldsymbol{v}}v$ and

$$\frac{\mathrm{d}\boldsymbol{p}_\parallel}{\mathrm{d}t} = m\,\gamma\left(1+\gamma^2\beta\,\frac{v}{c}\right)\frac{\mathrm{d}\boldsymbol{v}_\parallel}{\mathrm{d}t} = m\gamma^3\,\frac{\mathrm{d}\boldsymbol{v}_\parallel}{\mathrm{d}t}\,.$$

On the other hand, if the force is directed normal to the particle propagation we have $\dot{v} = 0$ and (1.18) reduces to

$$\frac{\mathrm{d}\boldsymbol{p}_\perp}{\mathrm{d}t} = m\,\gamma\,\frac{\mathrm{d}\boldsymbol{v}_\perp}{\mathrm{d}t}\,.$$

- Beam dynamics drastically different for parallel and perpendicular acceleration!
- Negligible radiation for parallel acceleration at high energy

Maxwell’s Equations

$$\nabla \cdot \boldsymbol{D} = \rho$$
$$\nabla \cdot \boldsymbol{B} = 0$$
$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$
$$\nabla \times \boldsymbol{H} = \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t}$$

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E}$$
$$\boldsymbol{B} = \mu_0 \boldsymbol{H}$$

$$c = (\epsilon_0 \mu_0)^{-1/2}$$

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377 \text{ Ohm}$$

- Wave equation

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \nabla^2 \right) \boldsymbol{E} = -\frac{1}{\epsilon_0} \left(\frac{\partial \boldsymbol{j}}{\partial t} + c^2 \nabla \rho \right)$$

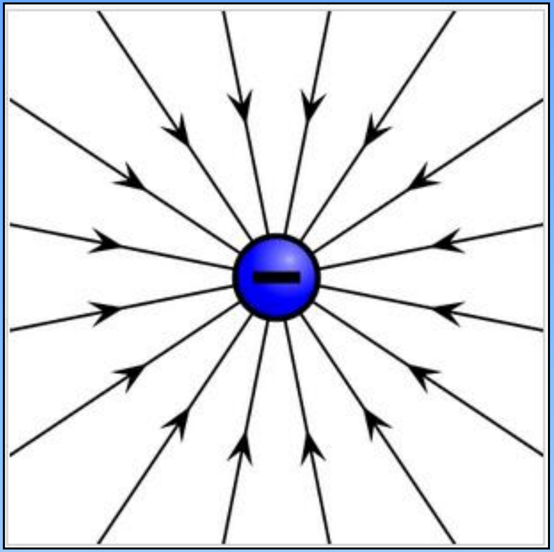
- Lorentz transformation of fields

$$E_z = E'_z\,, \qquad \boldsymbol{E}_\perp = \gamma \left(\boldsymbol{E}'_\perp - \boldsymbol{v} \times \boldsymbol{B}' \right)\,,$$
$$B_z = B'_z\,, \qquad \boldsymbol{B}_\perp = \gamma \left(\boldsymbol{B}'_\perp + \frac{1}{c^2} \boldsymbol{v} \times \boldsymbol{E}' \right)$$

Field of a moving electron

- In electron's frame, Coulomb field is

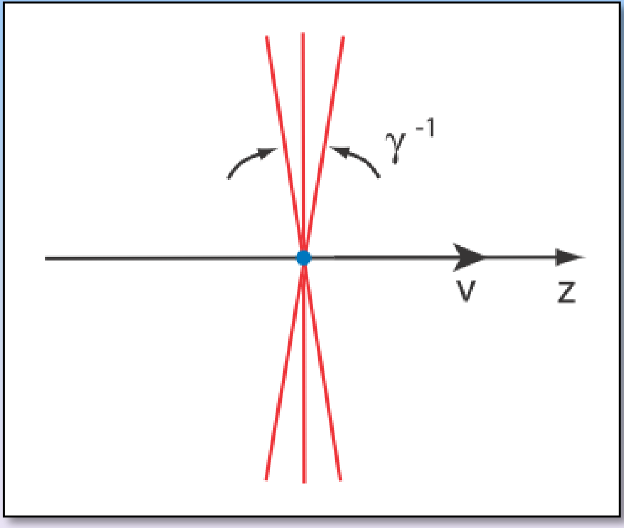
$$\boldsymbol{E}' = \frac{1}{4\pi\epsilon_0}\frac{e\boldsymbol{r}'}{r'^3}$$



- In lab frame, space charge fields are

$$E_x = \frac{1}{4\pi\epsilon_0}\frac{e\gamma x}{[x^2+y^2+\gamma^2(z-vt)^2]^{3/2}}$$
$$E_y = \frac{1}{4\pi\epsilon_0}\frac{e\gamma y}{[x^2+y^2+\gamma^2(z-vt)^2]^{3/2}}$$
$$E_z = \frac{1}{4\pi\epsilon_0}\frac{e\gamma (z-vt)}{[x^2+y^2+\gamma^2(z-vt)^2]^{3/2}}$$

$$\boldsymbol{B} = \frac{1}{c^2}\boldsymbol{v}\times\boldsymbol{E}$$



Lorentz Force

- Lorentz force

$$\boxed{\boldsymbol{F} = e\boldsymbol{E} + e\boldsymbol{v} \times \boldsymbol{B}}$$

- Momentum and energy change

$$\Delta \boldsymbol{p} = \int \boldsymbol{F} dt$$

$$\Delta E = \int \boldsymbol{F} d\boldsymbol{s}$$

$$\boxed{d\boldsymbol{s} = \boldsymbol{v} dt}$$

- **Energy exchange through *E* field only**

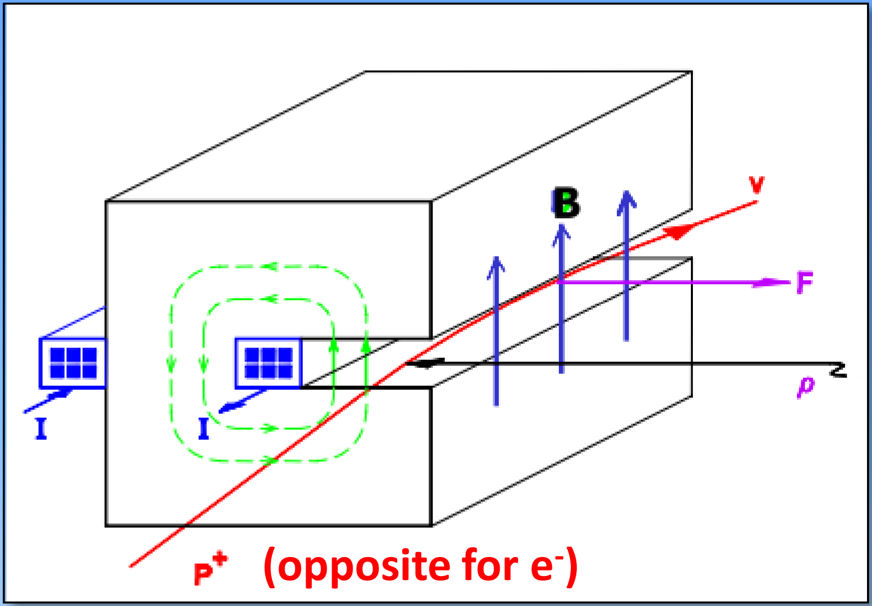
$$\Delta E = \int \boldsymbol{F} d\boldsymbol{s} = e \int \boldsymbol{E} \cdot d\boldsymbol{s} + e \underbrace{\int (\boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v} dt}_{= 0}$$

No work done by magnetic field!

Guiding beams: dipole

- Lorentz force

$$\boxed{\boldsymbol{F} = e\cancel{\boldsymbol{E}} + e\boldsymbol{v} \times \boldsymbol{B}}$$



- Centrifugal force

$$\boxed{F_{cf} = \frac{\gamma m c^2 \beta^2}{\rho}}$$

- Bending radius is obtained by balance the forces

$$\boxed{\frac{1}{\rho} = \frac{eB}{\gamma \beta m c^2}}$$

$$\boxed{\frac{1}{\rho}\,[\mathrm{m}^{-1}] = 0.2998\frac{B[\mathrm{T}]}{\beta E[\mathrm{GeV}]}}$$

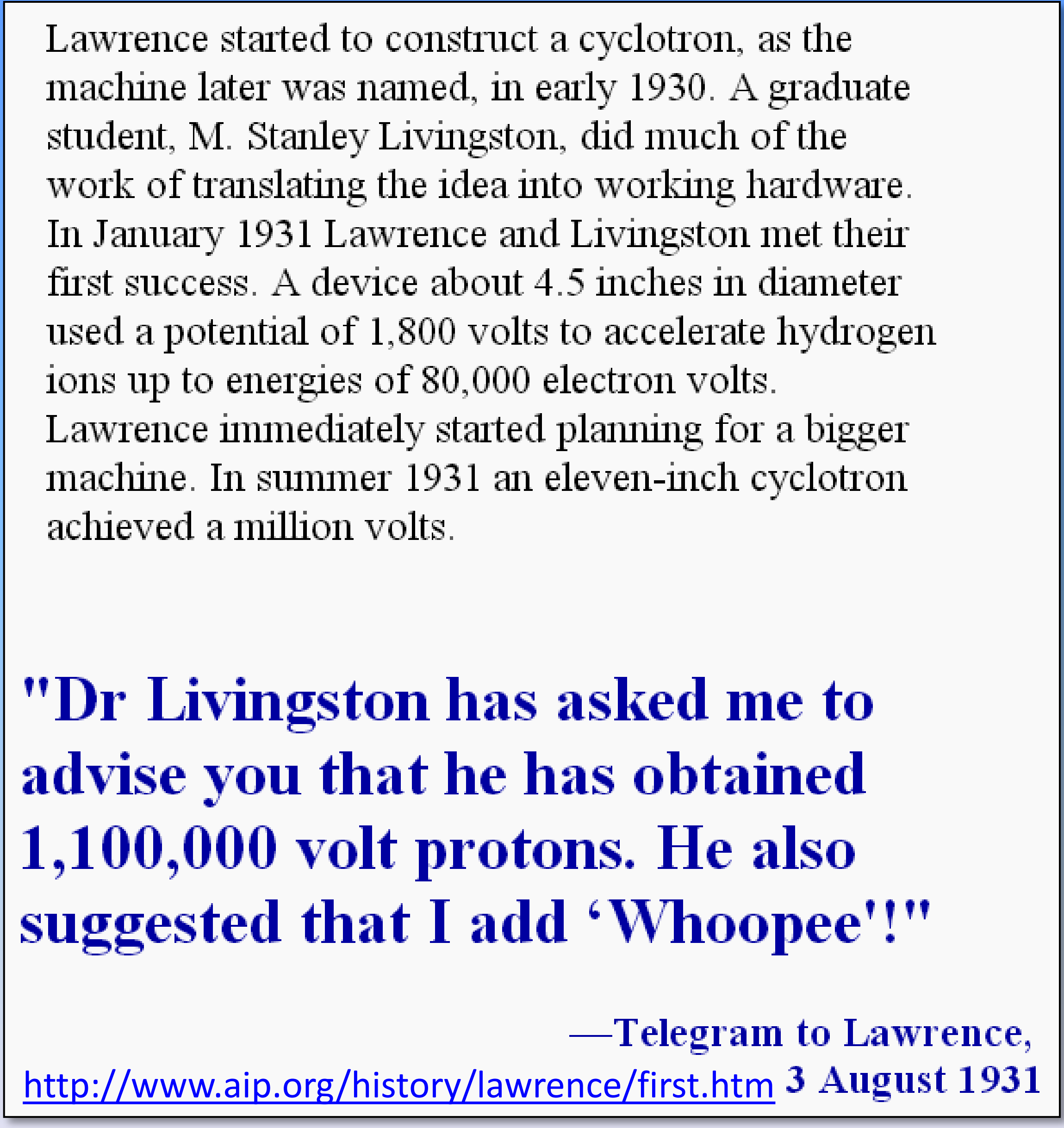
Cyclotron

- If beam moves circularly, re-traverses the same accelerating section again and again, we can accelerate the beam repetitively

Ernest O. Lawrence in 1930

The first cyclotron with a diameter of 5 inches

[Ref.]: Photography gallery of Lawrence Berkeley National Laboratory,
http://cso.lbl.gov/photo/gallery/



Lawrence started to construct a cyclotron, as the machine later was named, in early 1930. A graduate student, M. Stanley Livingston, did much of the work of translating the idea into working hardware. In January 1931 Lawrence and Livingston met their first success. A device about 4.5 inches in diameter used a potential of 1,800 volts to accelerate hydrogen ions up to energies of 80,000 electron volts. Lawrence immediately started planning for a bigger machine. In summer 1931 an eleven-inch cyclotron achieved a million volts.

"Dr Livingston has asked me to advise you that he has obtained 1,100,000 volt protons. He also suggested that I add 'Whoopee'!"

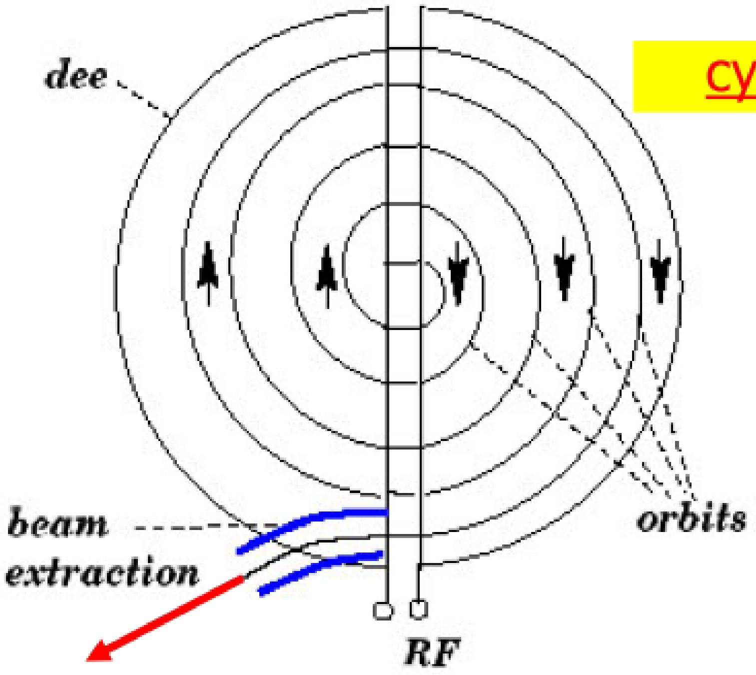
—Telegram to Lawrence,
http://www.aip.org/history/lawrence/first.htm **3 August 1931**

Lawrence was my teacher when I built the first cyclotron. He got a Nobel prize for it. I got a Ph.D. (- S. Livingston, years later)

From Cyclotron to Synchrotron

- Cyclotron does not work for relativistic beams.

cyclotron

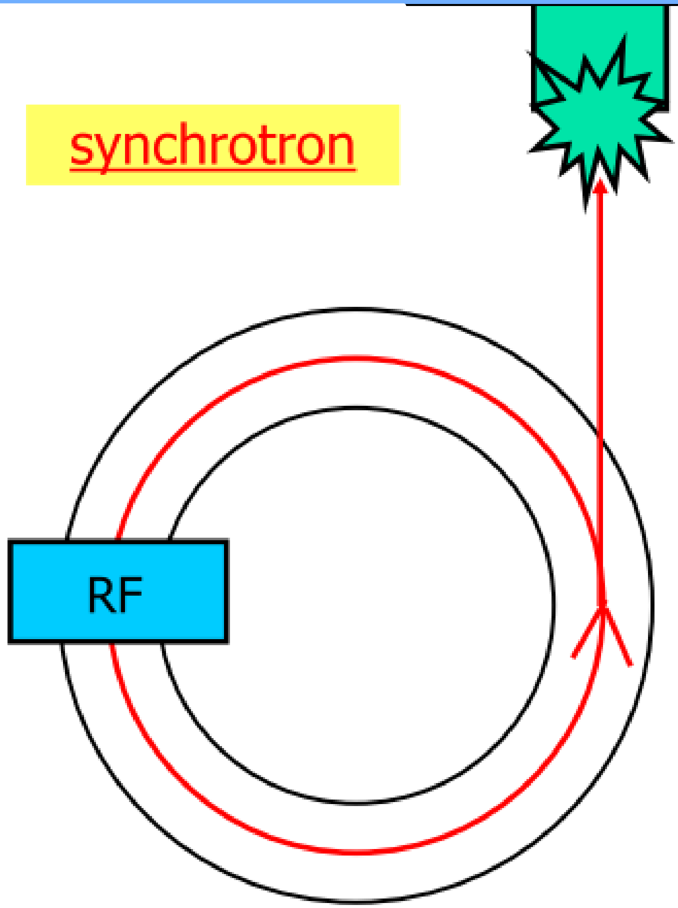


huge dipole, compact design,

B = constant

low energy, single pass.

synchrotron

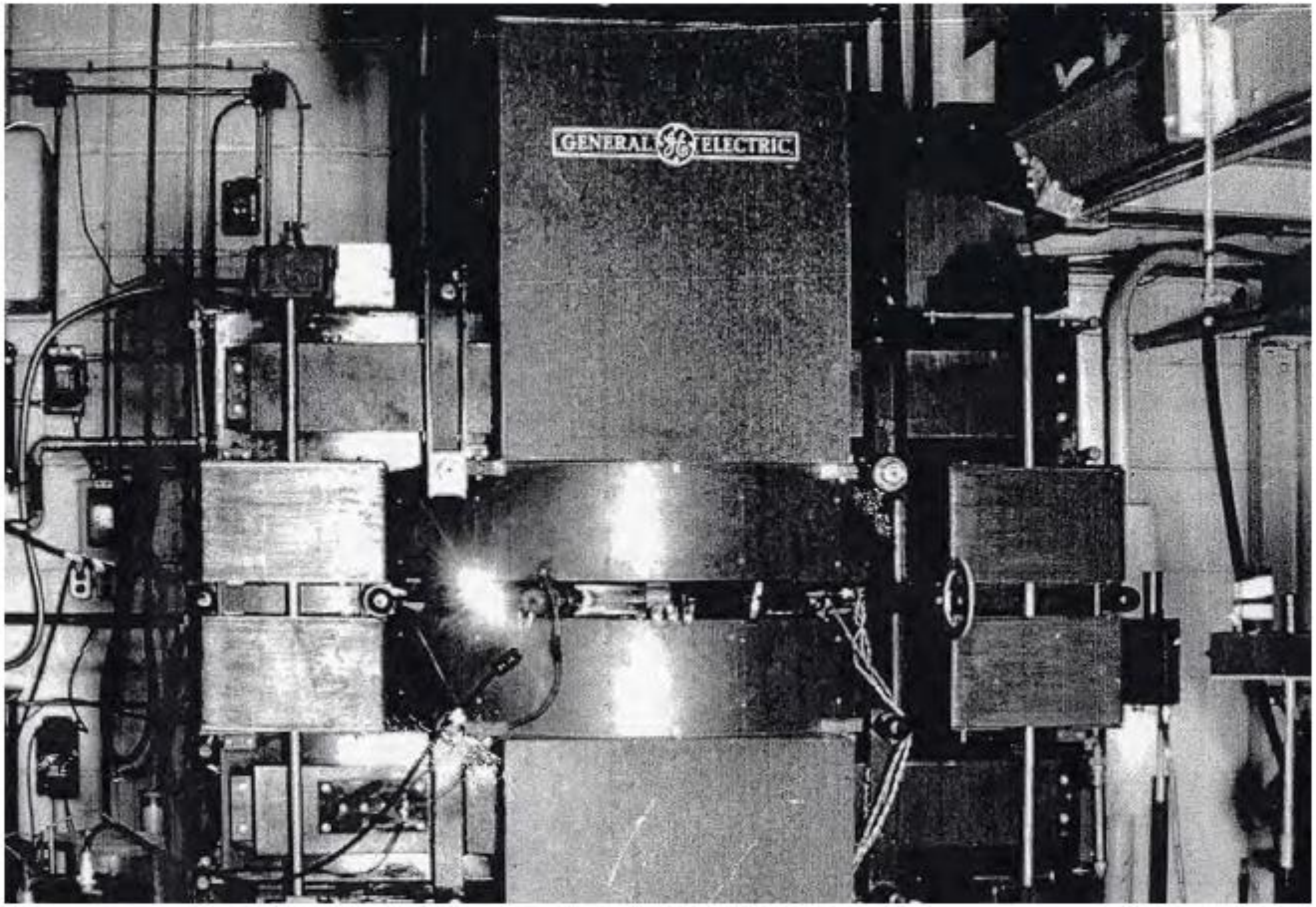


varying B, small magnets, high energy

Synchrotron

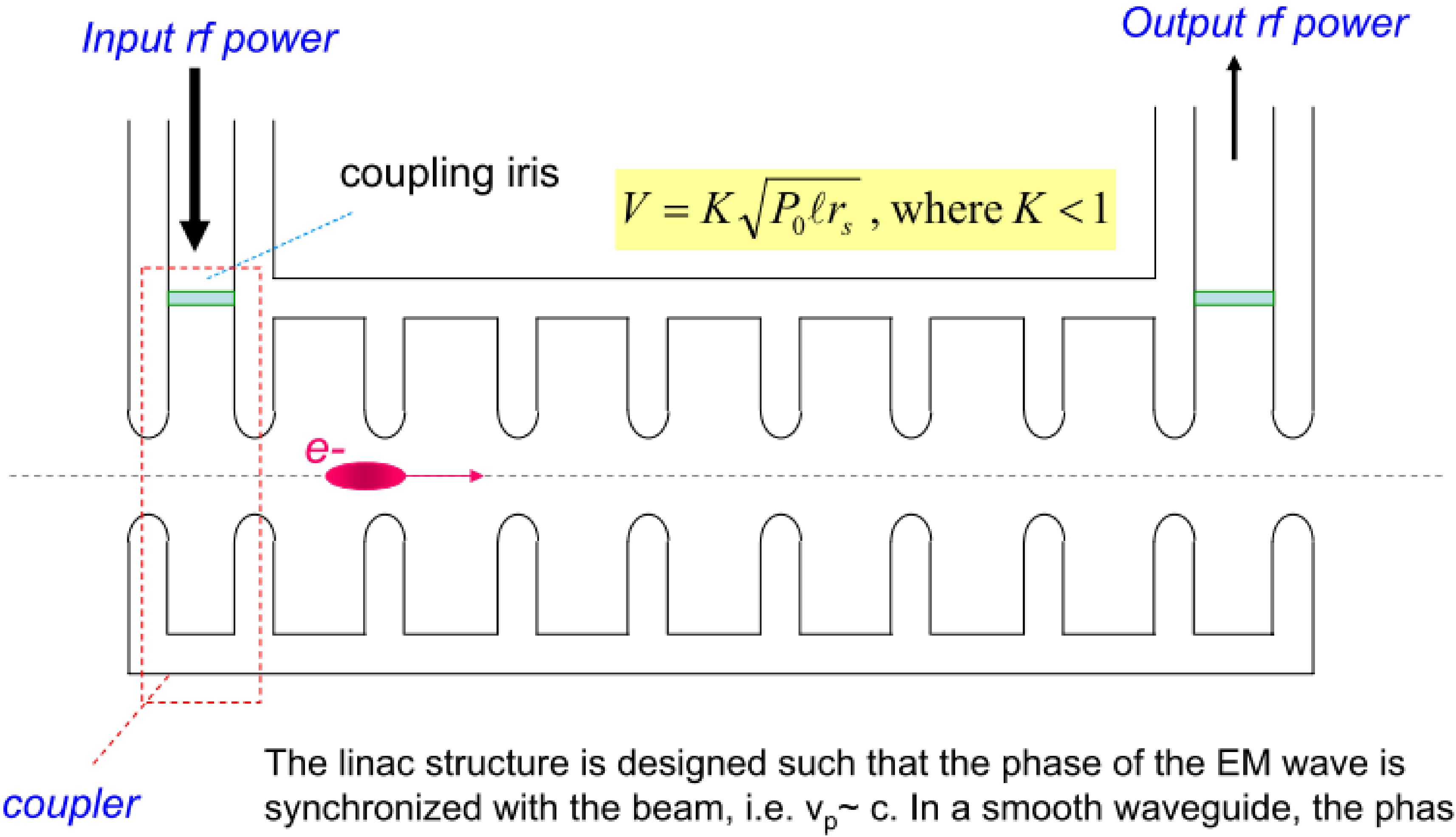
- GE synchrotron observed first synchrotron radiation (1946) and opened a new era of accelerator-based light sources.

The first purpose-built synchrotron to operate was built with a glass vacuum chamber



Electron linac

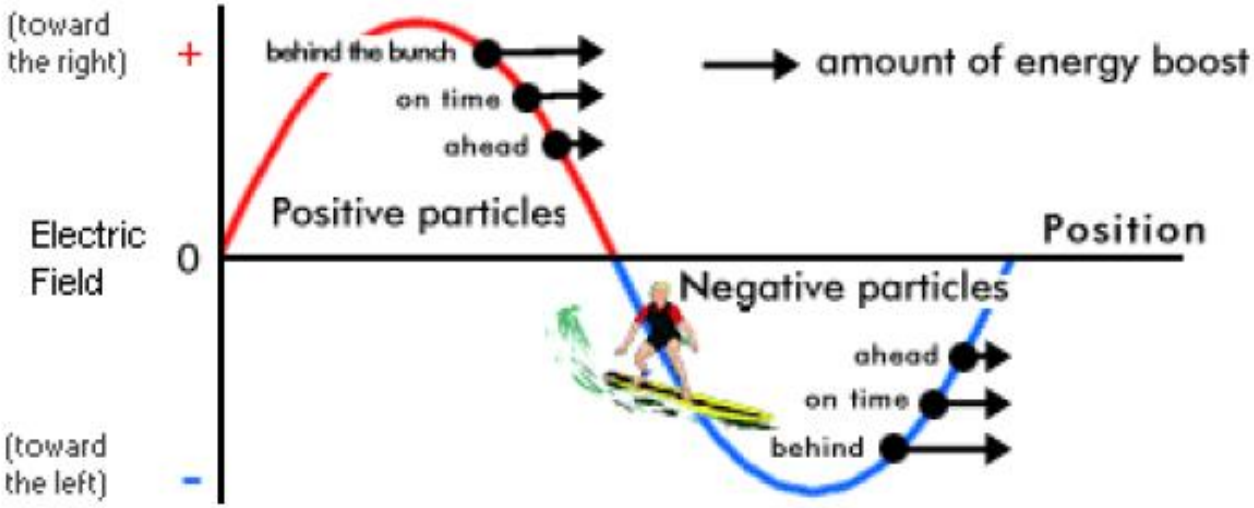
The rf energy is used to launch a traveling wave or standing wave in an array of cavities.

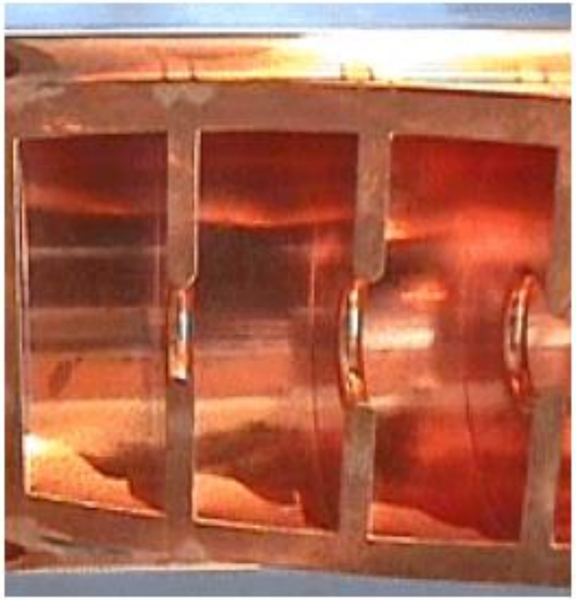


$$V = K\sqrt{P_0 \ell r_s}\ \text{, where } K < 1$$

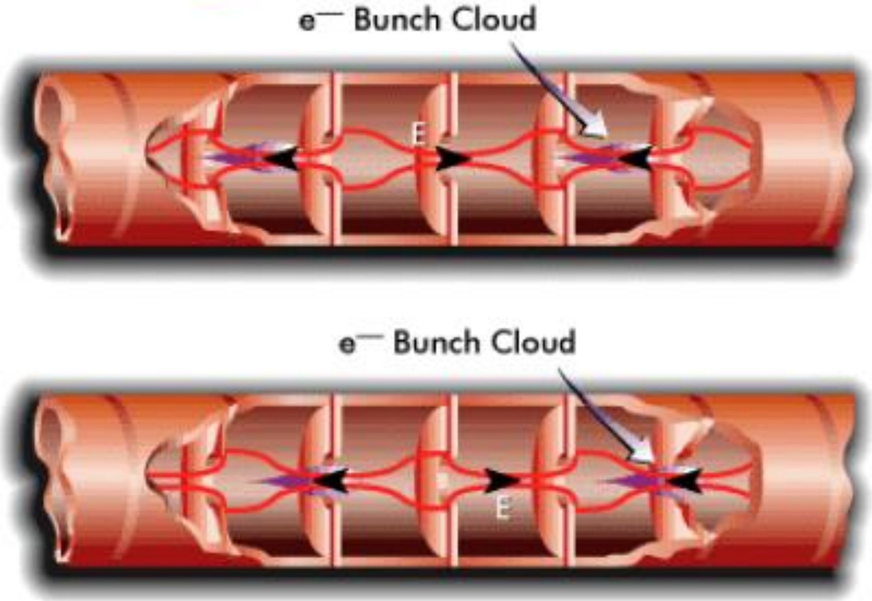
The linac structure is designed such that the phase of the EM wave is synchronized with the beam, i.e. $v_p \sim c$. In a smooth waveguide, the phase velocity $v_p > c$. Those disks are used to slow down the waveguide phase velocity in order to achieve synchronism with the electron beams.

Electron Linac (disk loaded structure)





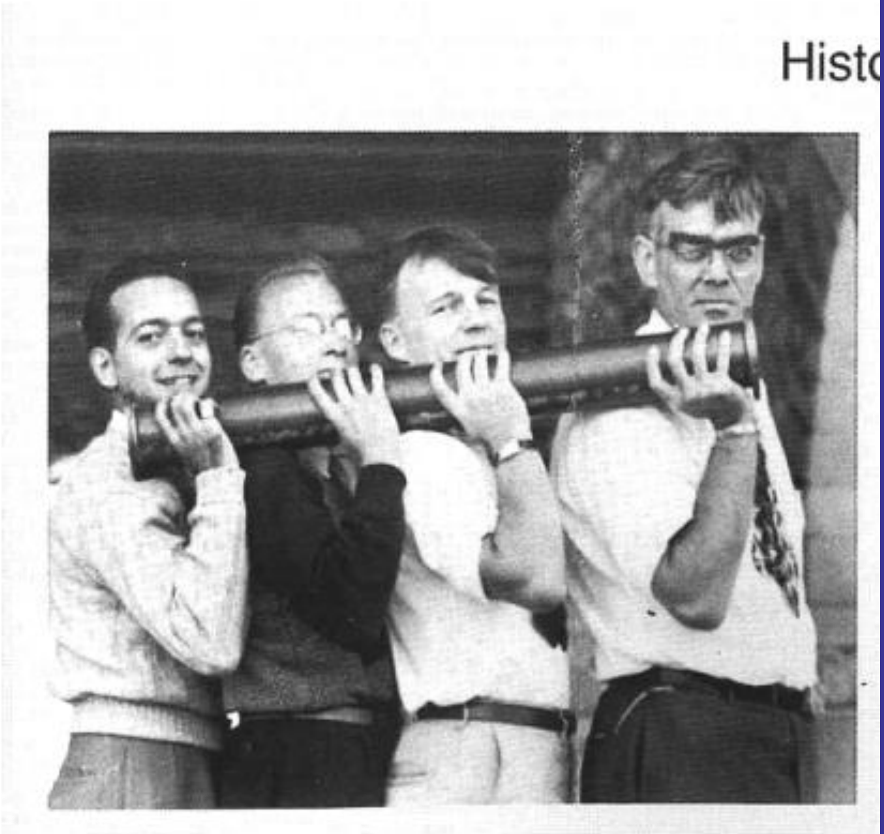
[Ref.] http://www.slac.stanford.edu

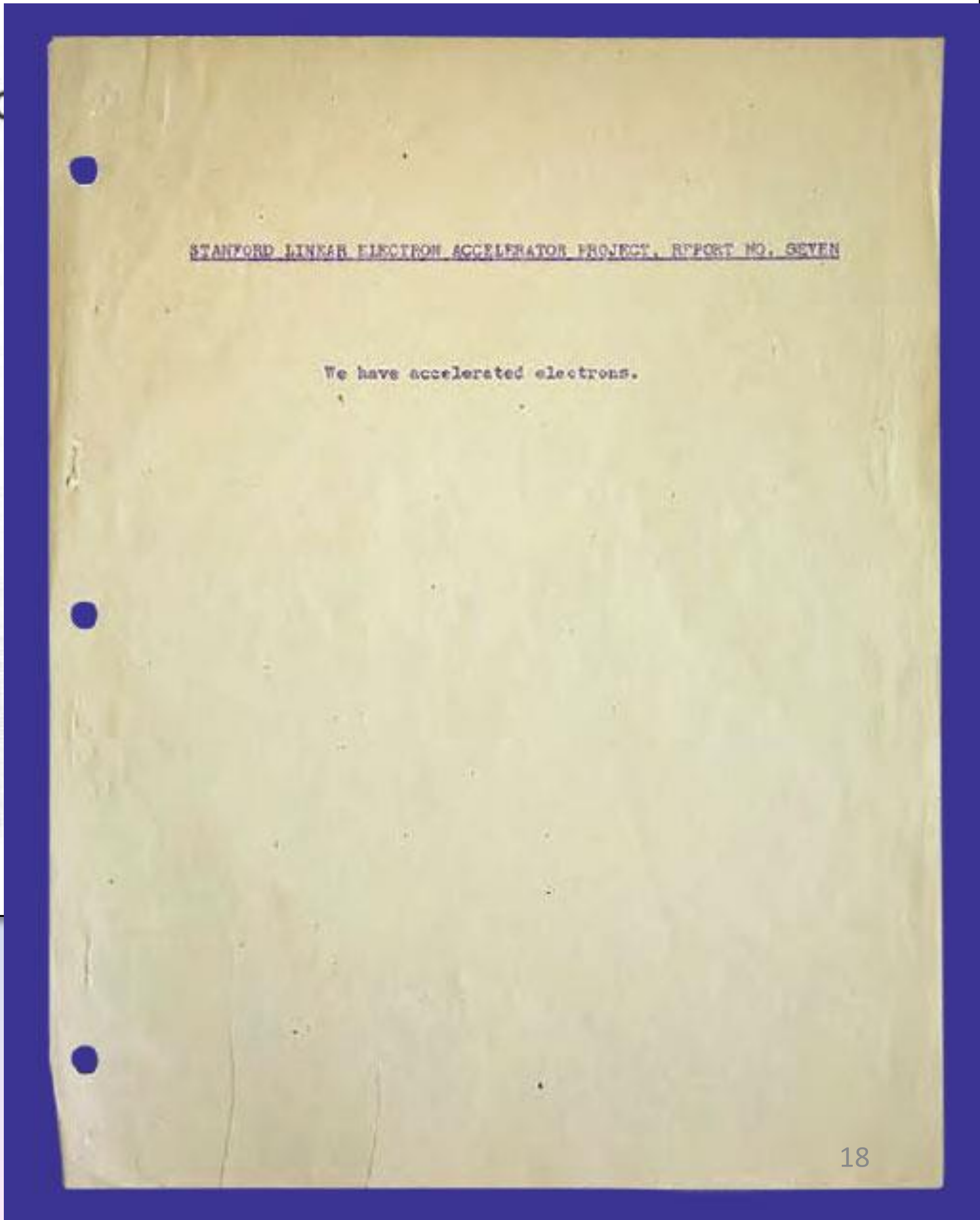


1/20,000,000,000 second later

(notice how far the bunches have moved)

Disk loaded structure made at Stanford Univ. (1947)

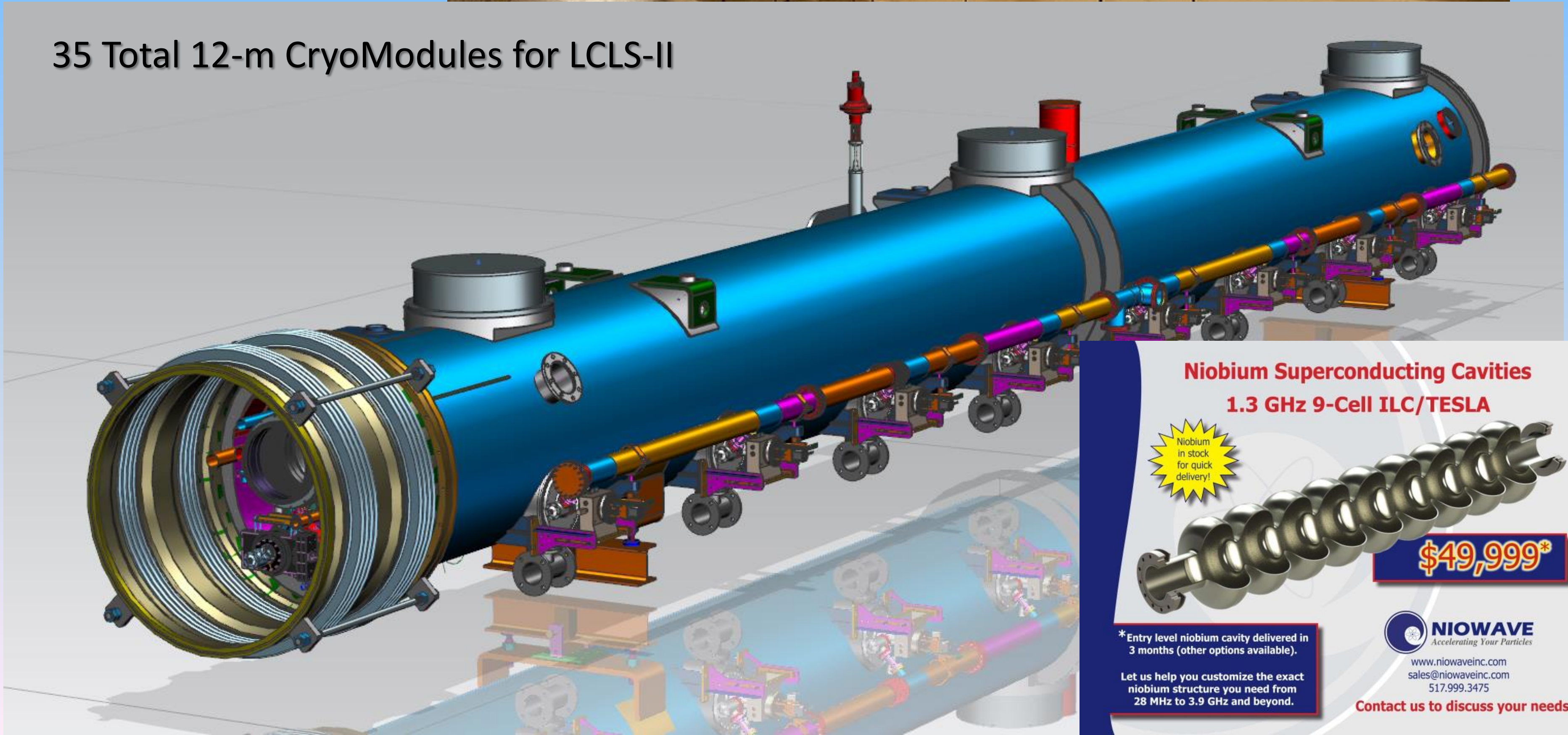




<u>STANFORD LINEAR ELECTRON ACCELERATOR PROJECT, REPORT NO. SEVEN</u>

We have accelerated electrons.

SLAC linac



35 Total 12-m CryoModules for LCLS-II

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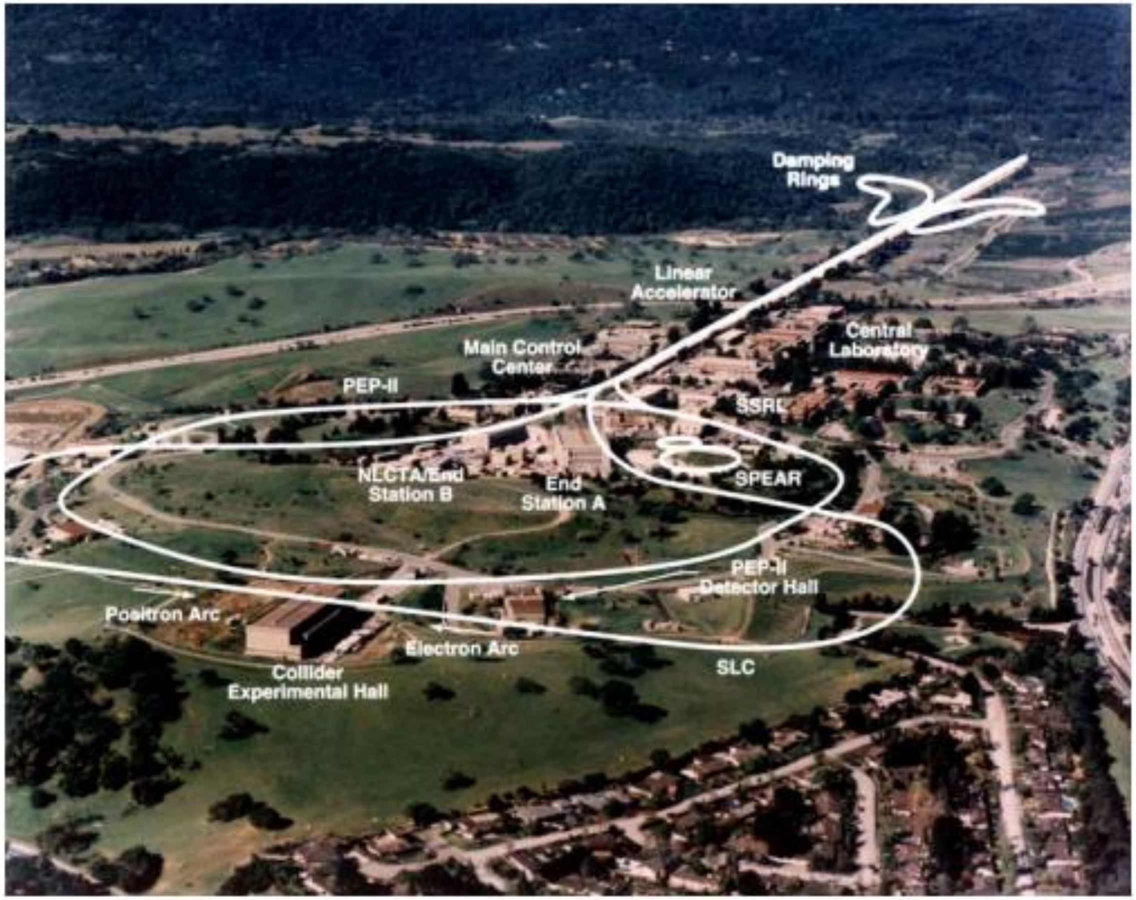
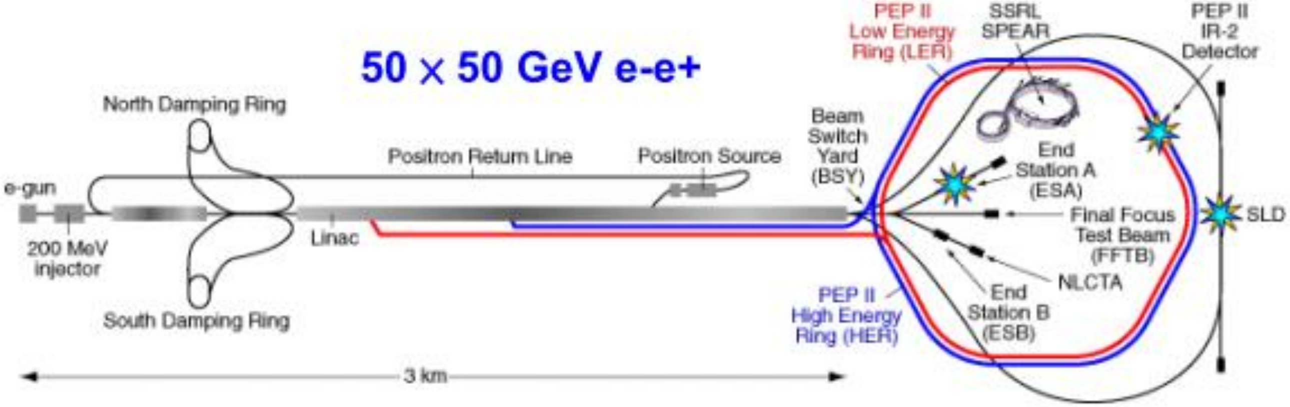
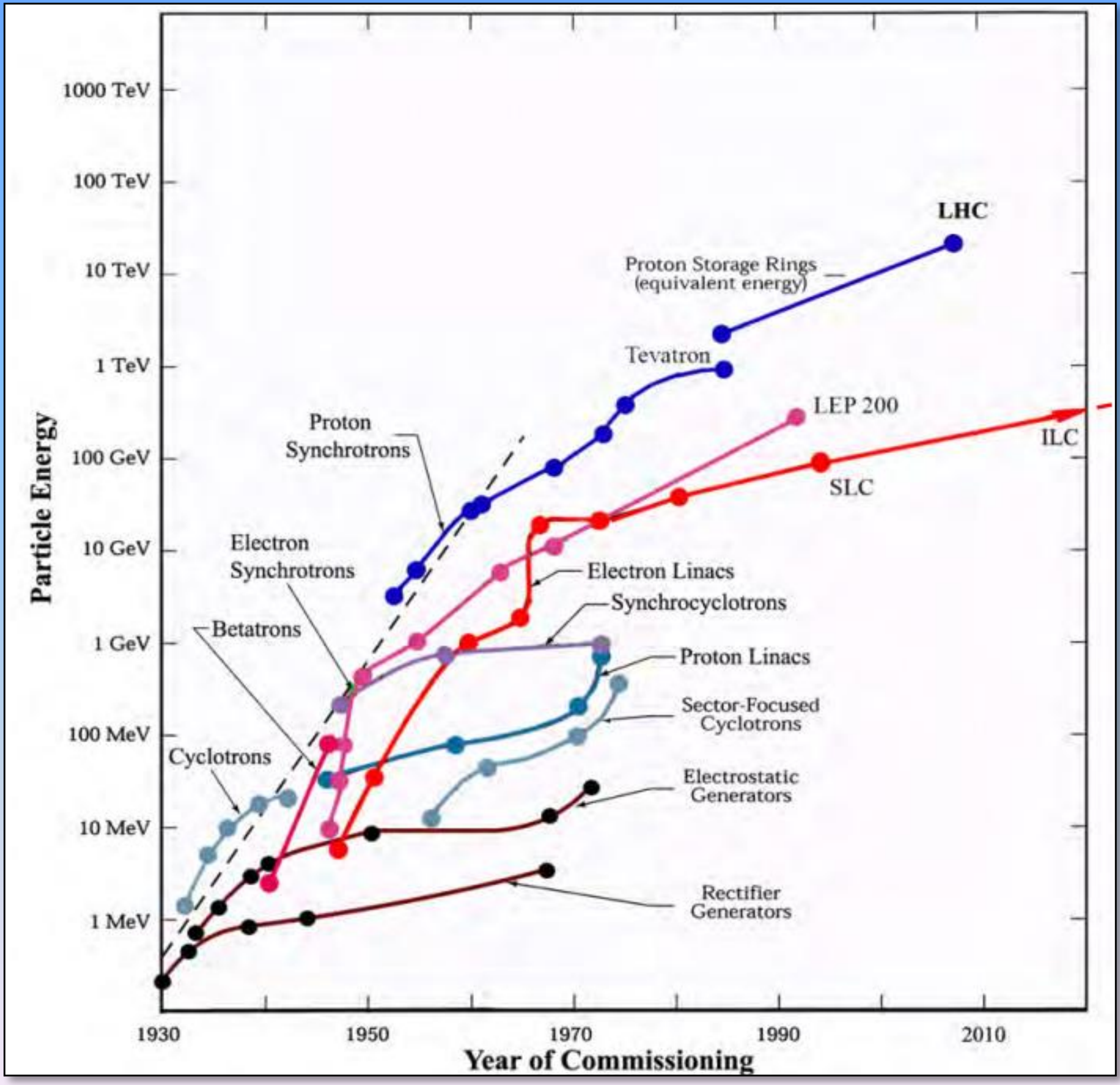


50 × 50 GeV e-e+



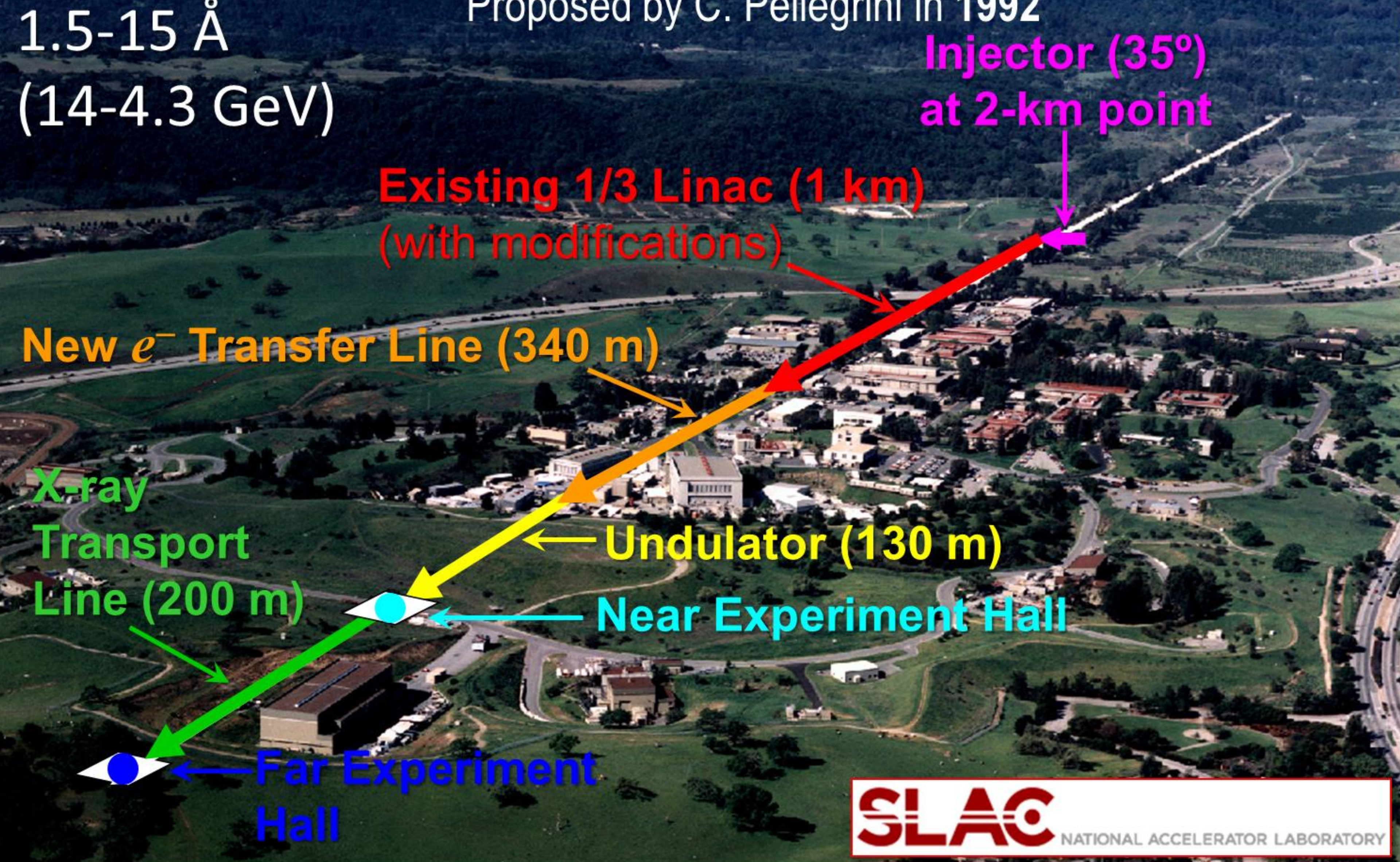
Livingston Plot for High-Energy Accelerators



Linac **C**oherent **L**ight **S**ource (**LCLS**) at *SLAC*

X-FEL based on last 1-km of existing 3-km linac

Proposed by C. Pellegrini in **1992**



1.5-15 Å
(14-4.3 GeV)

**Injector (35º)
at 2-km point**

Existing 1/3 Linac (1 km)
(with modifications)

New e^- Transfer Line (340 m)

Undulator (130 m)

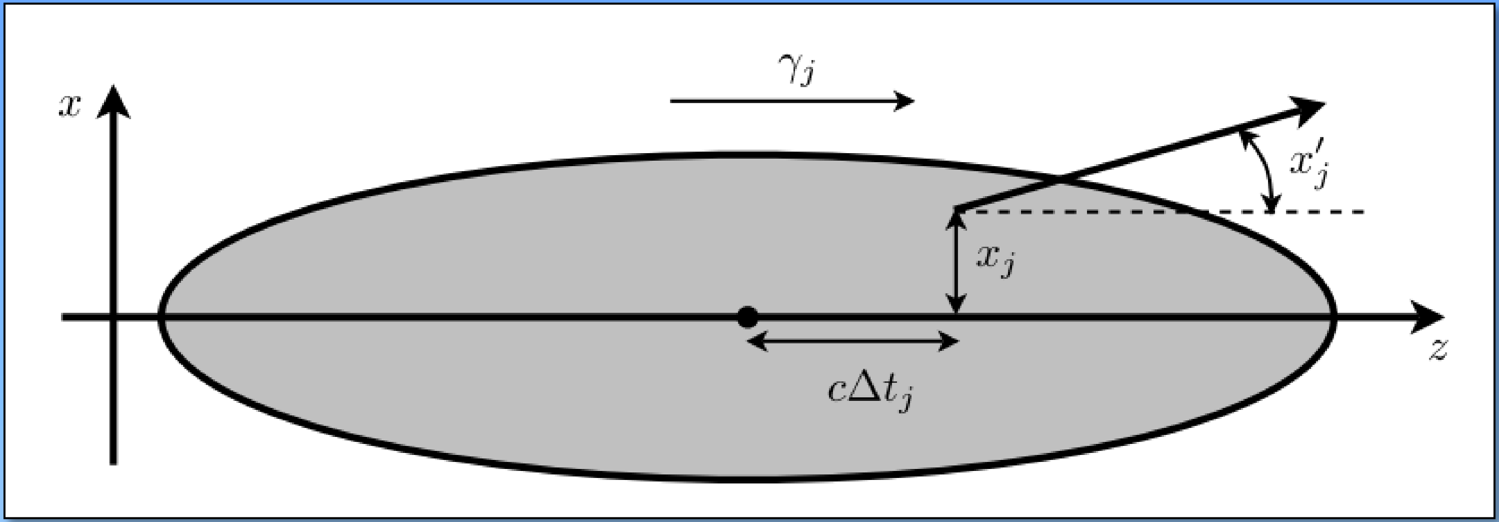
**X-ray
Transport
Line (200 m)**

Near Experiment Hall

**Far Experiment
Hall**

SLAC NATIONAL ACCELERATOR LABORATORY

Beam description



- Beam phase space (*x*,*x'*,*y*,*y'*,*Δt*, *Δγ*)

$$x' \equiv \frac{dx}{dz} = \frac{dx/dt}{dz/dt} = \frac{1}{v_z}\frac{dx}{dt}$$

$$\Delta\gamma_j \equiv \gamma_j - \gamma_0$$

- Consider paraxial beams such that

$$|\boldsymbol{x}'| = \sqrt{x'^2 + y'^2} \approx \frac{1}{c}\sqrt{v_x^2 + v_y^2} \ll 1$$

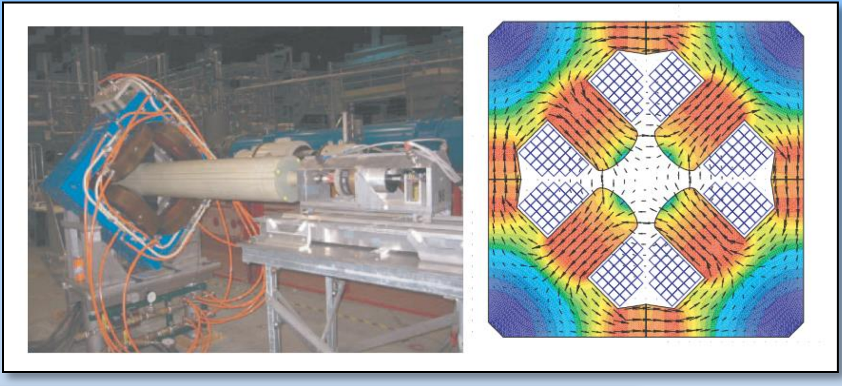
Linear optics for beam transport

- Transport matrix

$$\begin{bmatrix} x \\ x' \end{bmatrix}_{\mathrm{o}} = \mathsf{M}(z_{\mathrm{i}}, z_{\mathrm{o}}) \begin{bmatrix} x \\ x' \end{bmatrix}_{\mathrm{i}}$$

- Free space drift

$$\begin{bmatrix} x \\ x' \end{bmatrix}_{\mathrm{o}} = \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}_{\mathrm{i}} \equiv \mathsf{M}_{\ell} \begin{bmatrix} x \\ x' \end{bmatrix}_{\mathrm{i}}$$



- Quadrupole (de-)focusing

$$\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}_{\mathrm{o}} = \mathsf{M}_f \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}_{\mathrm{i}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/f & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 1 \end{bmatrix} \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}_{\mathrm{i}}$$

Beam properties

- Second moments of beam distribution

rms size

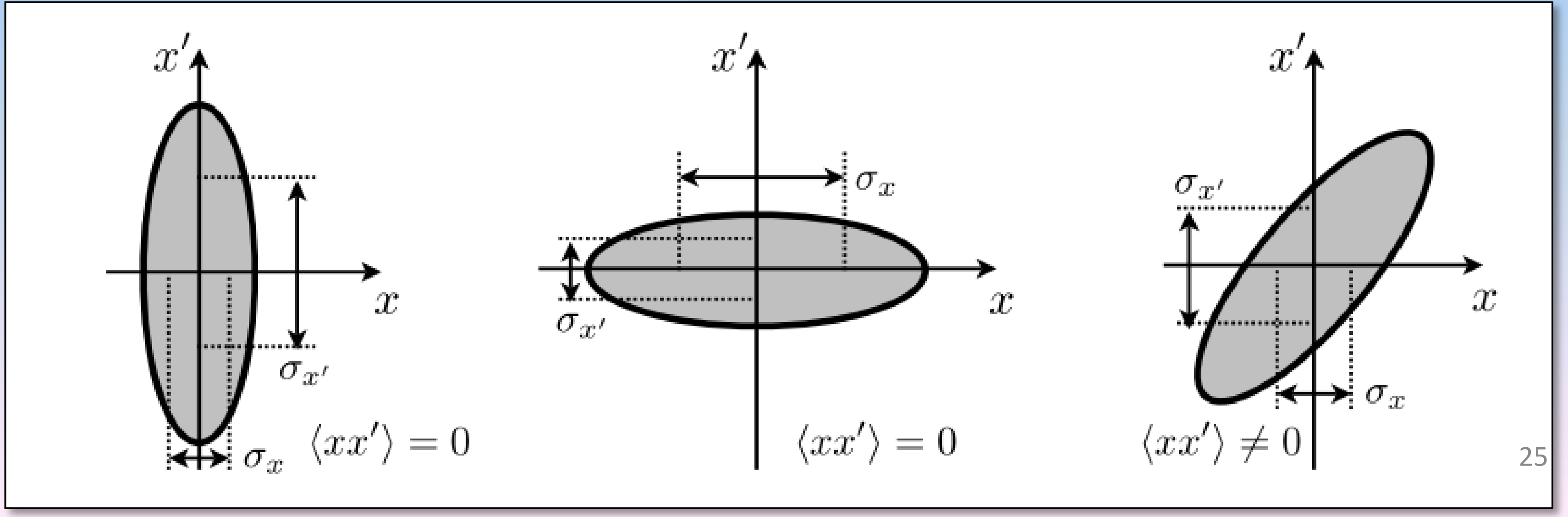
$$\sigma_x^2(z) = \langle x^2 \rangle = \frac{1}{N_e} \sum_j x_j^2 .$$

rms divergence

$$\sigma_{x'}^2(z) = \left\langle x'^2 \right\rangle = \frac{1}{N_e} \sum_j x_j'^2 .$$

correlation

$$\langle x\, x' \rangle = \frac{1}{N_e} \sum_j x_j x_j' .$$



Beam emittance

- **Emittance or geometric emittance**

$$\varepsilon_x \equiv \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle}$$

- Emittance is **conserved** in a **linear** transport system

- **Normalized emittance** is conserved in a linear system including acceleration

$$\varepsilon_{x,n} = \beta_z\gamma\varepsilon_x \approx \gamma\varepsilon_x$$

- Normalized emittance is hence an important figure of merit for electron sources

- Preservation of emittances is critical for accelerator designs.

Beam optics function

- Optics functions (Twiss parameters)

$$\beta_x = \frac{\langle x^2 \rangle}{\varepsilon_x} \qquad \gamma_x = \frac{\langle x'^2 \rangle}{\varepsilon_x} \qquad \alpha_x = -\frac{\langle x x' \rangle}{\varepsilon_x}$$

$$\beta_x \gamma_x - \alpha_x^2 = 1$$

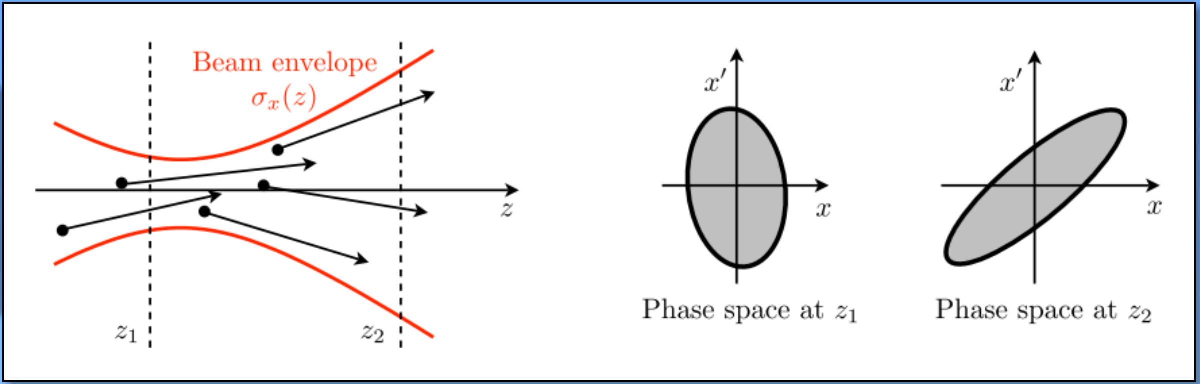
- Given beta function along beamline

$$\sigma_x(z) = \sqrt{\varepsilon_x \beta_x(z)}$$

Free space propagation

- Single particle

$$\begin{bmatrix} x \\ x' \end{bmatrix}_{\mathrm{o}} = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}_{\mathrm{i}}$$



- Beam envelope

$$\langle x_{\mathrm{o}}^2\rangle = \left\langle \left(x_{\mathrm{i}} + z\,x'_{\mathrm{i}}\right)^2 \right\rangle = \langle x_{\mathrm{i}}^2\rangle + 2z\,\langle x_{\mathrm{i}}x'_{\mathrm{i}}\rangle + z^2\langle x'^2_{\mathrm{i}}\rangle$$

$$\beta_x(z) = \beta_x(0) + z^2\gamma_x(0)$$

$$\beta_x(z) = \beta_x^* + \frac{z^2}{\beta_x^*}$$

$$\sigma_x(z) = \sqrt{\varepsilon_x\beta_x(z)} = \sqrt{\varepsilon_x\left(\beta_x^* + \frac{z^2}{\beta_x^*}\right)}$$

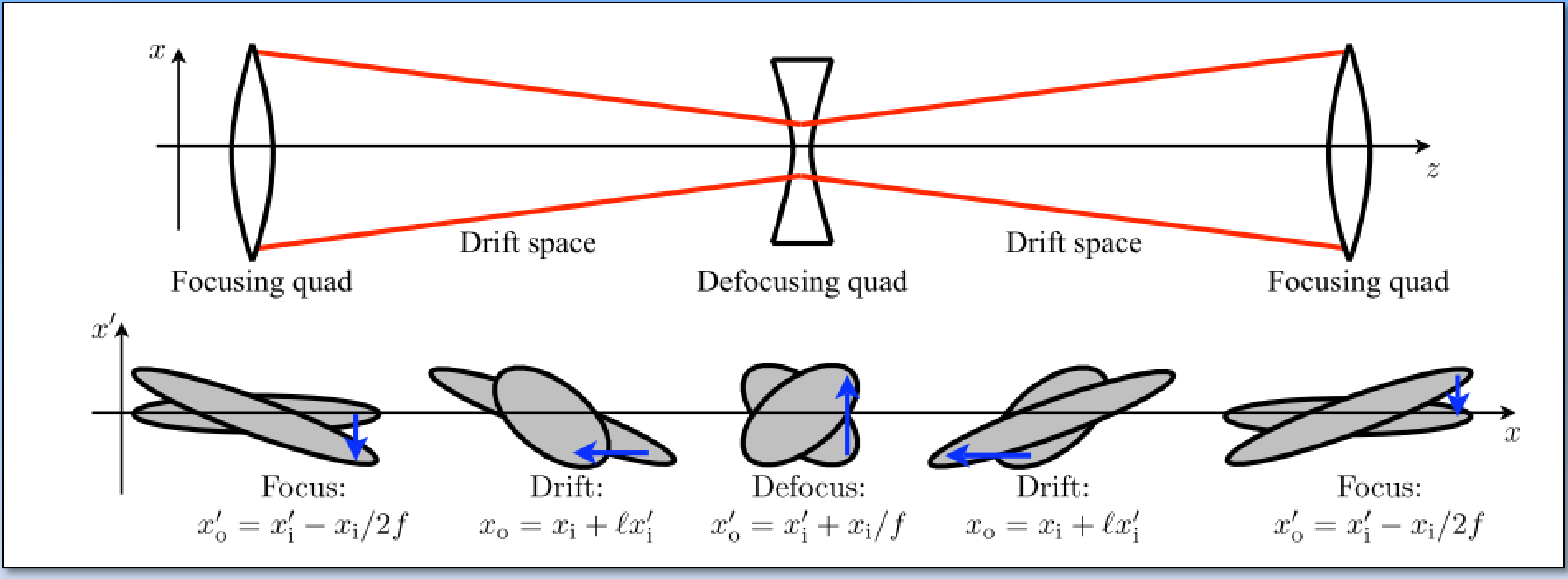
- Analogous with Gaussian laser beam

$$\varepsilon_x \leftrightarrow \frac{\lambda}{4\pi} \qquad \beta_x^* \leftrightarrow Z_R.$$

FODO lattice

- Multiple elements (e. g., FODO lattice)

$$\mathsf{M} = \mathsf{M}_N\,\mathsf{M}_{N-1}\,\ldots\,\mathsf{M}_2\,\mathsf{M}_1$$



$$\begin{aligned}\mathsf{M}_{\mathrm{FODO}} &= \begin{bmatrix} 1 & 0 \\ -1/2f & 1 \end{bmatrix} \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/2f & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 - \frac{\ell^2}{2f^2} & 2\ell\left(1 + \frac{\ell}{2f}\right) \\ -\frac{\ell}{2f^2}\left(1 - \frac{\ell}{2f}\right) & 1 - \frac{\ell^2}{2f^2} \end{bmatrix}.\end{aligned}$$

FODO lattice II

For periodic motion we have $\beta_x(0) = \beta_x(2\ell)$ and $\gamma_x(0) = \gamma_x(2\ell)$, while vanishing correlation α_x at the two planes implies that $\beta_x(0) = 1/\gamma_x(0)$

- Maximum beta

$$\beta_x(0) = 2\sqrt{\frac{2f^3 + f^2\ell}{2f - \ell}} \approx 2\,|f|\left(1 + \frac{\ell}{2f}\right)$$

- Minimum beta

$$\beta_x(\ell) \approx 2\,|f|\left(1 - \frac{\ell}{2f}\right)$$

- When *f* >> *l*

$$\begin{array}{rcr}
\beta_x(z) \approx \bar{\beta}_x = 2f & \longrightarrow & \langle x^2 \rangle \approx 2\varepsilon_x f \\
\gamma_x(z) \approx \dfrac{2}{\bar{\beta}_x} = \dfrac{1}{f} & \longrightarrow & \langle x'^2 \rangle \approx \dfrac{\varepsilon_x}{f} \\
\alpha_x^2(z) \approx \bar{\beta}_x \bar{\gamma}_x - 1 = 1 & \longrightarrow & \langle x x' \rangle \approx \pm\varepsilon_x .
\end{array}$$

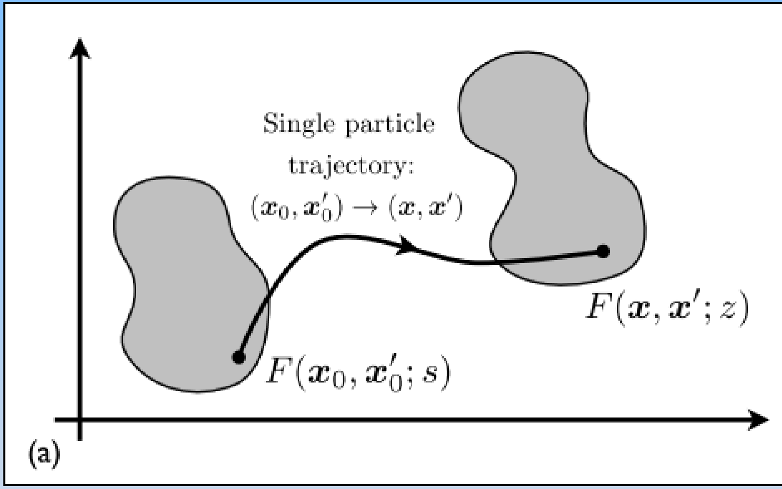
Electron distribution in phase space

- We define the distribution function *F* so that

$$\boxed{N_e F(\Delta t, \Delta\gamma, \boldsymbol{x}, \boldsymbol{x}'; z)\, d\boldsymbol{x} d\boldsymbol{x}' d(\Delta t) d(\Delta\gamma)}$$

is the number of electrons per unit phase space volume

- Since the number of electrons is an invariant function of *z*, distribution function satisfies **Liouville theorem**



$$\boxed{\frac{d}{dz}F = \left[\frac{\partial}{\partial z} + (\Delta t)'\frac{\partial}{\partial \Delta t} + (\Delta\gamma)'\frac{\partial}{\partial \Delta\gamma} + \boldsymbol{x}' \cdot \frac{\partial}{\partial \boldsymbol{x}} + \boldsymbol{x}'' \cdot \frac{\partial}{\partial \boldsymbol{x}'}\right] F = 0}$$

equations of motion

Gaussian beam distribution

- Represent the ensemble of electrons with a continuous distribution function (e.g., Gaussian in *x* and *x'*)

$$F(x,x';z)=\frac{1}{2\pi\varepsilon_x}\exp\left\{-\frac{1}{2\varepsilon_x}\left[\gamma_x(z)x^2+\beta_x(z)x'^2+2\alpha_x(z)xx'\right]\right\}$$

- For free space propagation

$$\beta_x(z)=\beta_x^*+\frac{z^2}{\beta_x^*}$$

$$F(x,x';z)=\frac{1}{2\pi\varepsilon_x}\exp\left[-\frac{(x-x'z)^2}{2\beta_x^*\varepsilon_x}-\frac{x'^2}{2\varepsilon_x/\beta_x^*}\right]$$

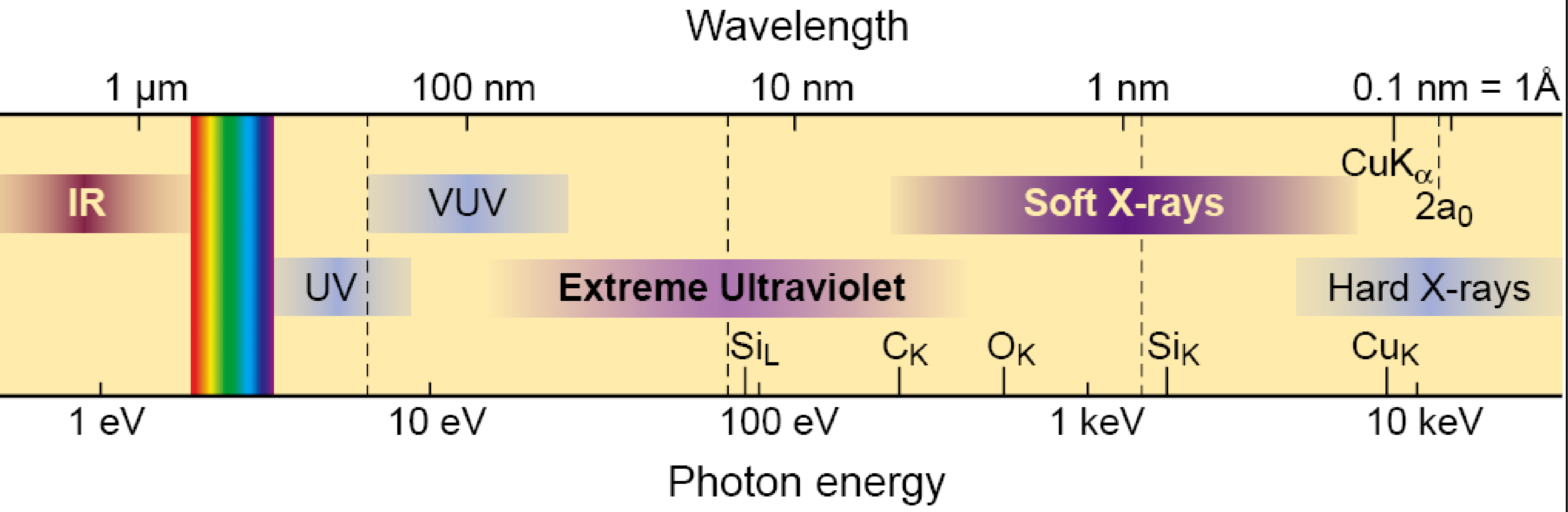
- Distribution in physical space can be obtained by integrating *F* over the angle

$$\int dx'\;F(x,x';z)=\frac{\exp\left[-\frac{x^2}{2\sigma_x^{*2}(1+z^2/\beta_x^{*2})}\right]}{\sqrt{2\pi}\,\sigma_x^*\sqrt{1+z^2/\beta_x^{*2}}}$$

Photon or radiation beams

- Introduction to radiation
- Radiation diffraction and emittance
- Coherence and Brightness
- Radiation intensity and bunching
- Accelerator based light sources

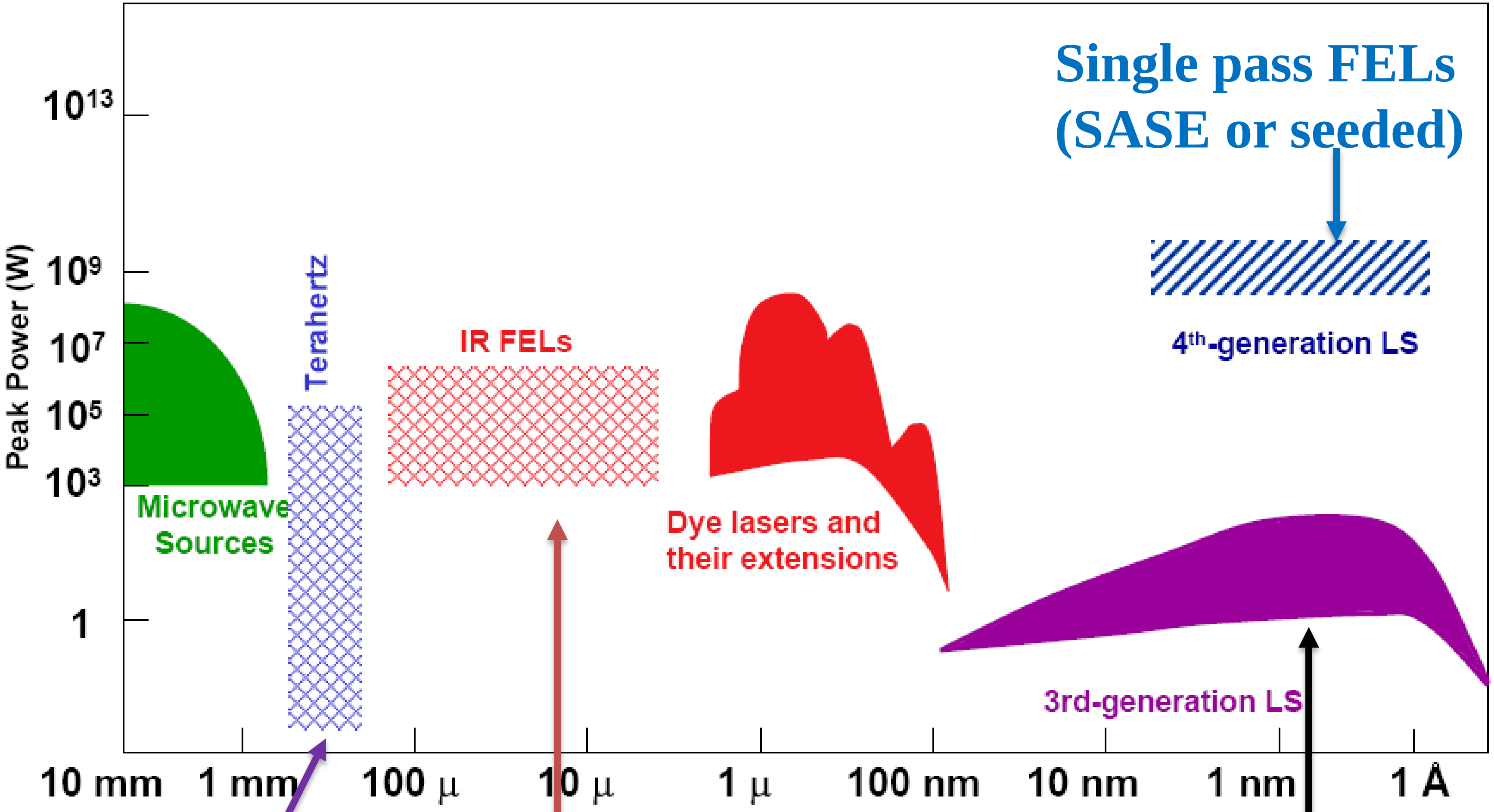
Photon wavelength and energy



- **See smaller features**
- **Write smaller patterns**
- **Elemental and chemical sensitivity**

$$\boxed{\hbar\omega \cdot \lambda = hc = 1239.842 \text{ eV nm}}$$

Opportunities for Tunable Source of Radiation



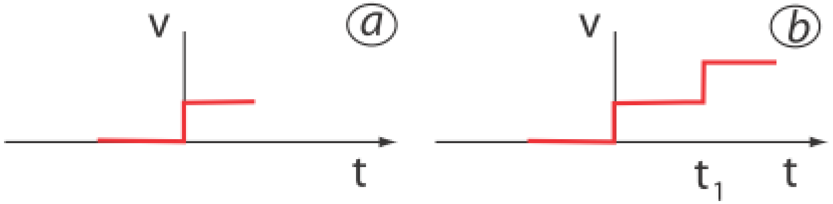
Single pass FELs
(SASE or seeded)

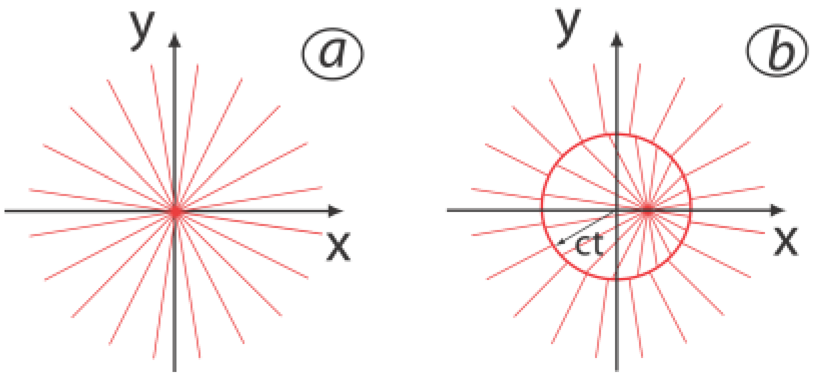
Various accelerator
and non-acc. sources

FEL oscillators
(High-average power)

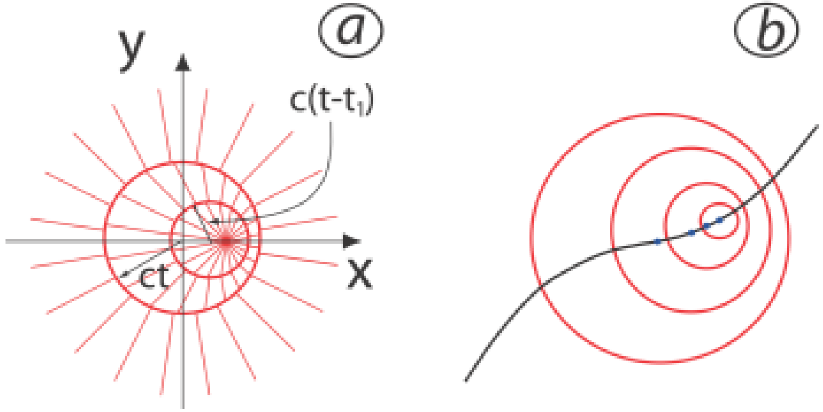
Synchrotron radiation
Undulator radiation

Radiation from Accelerated Electrons



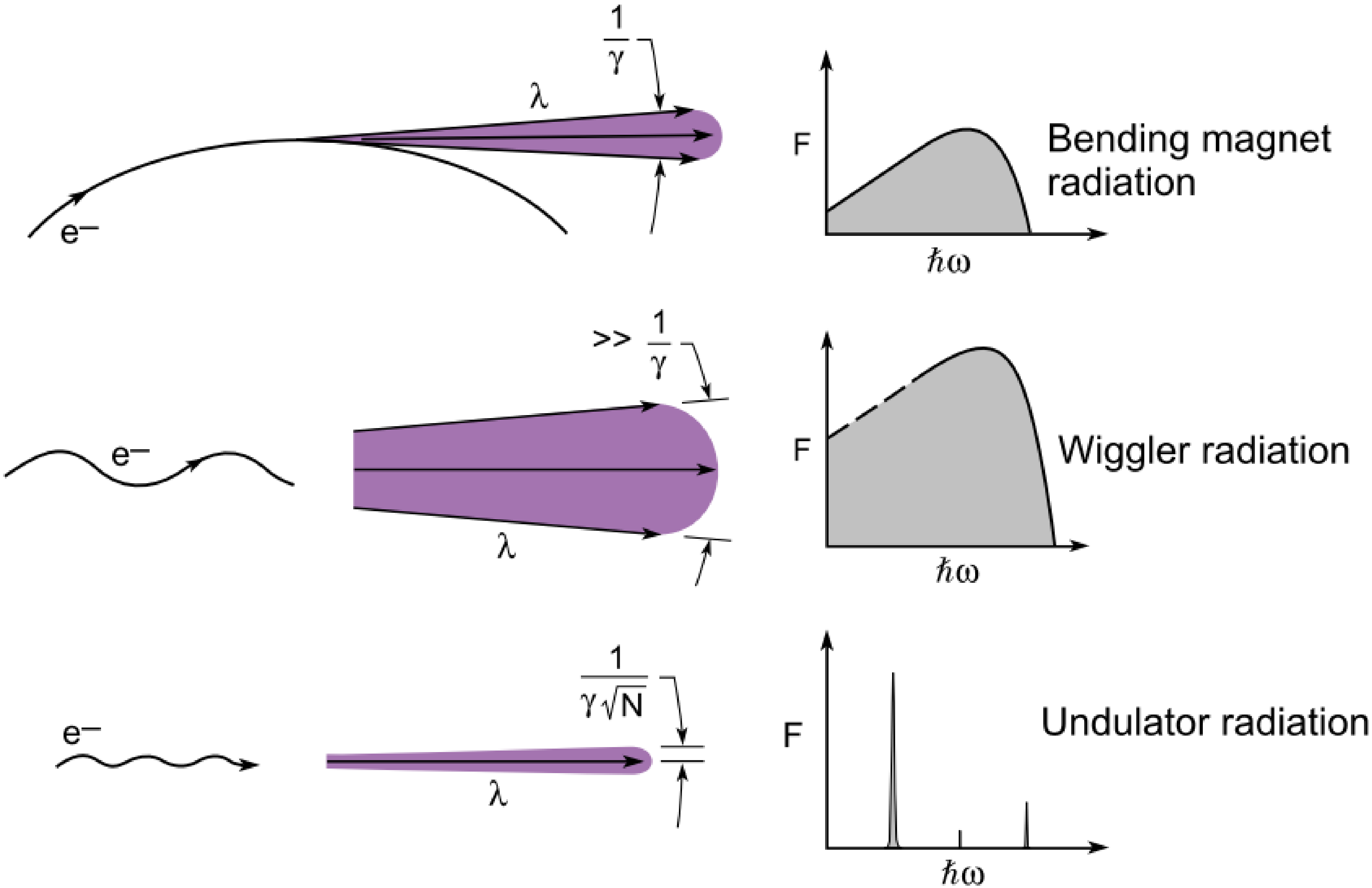


The field outside of the circle of radius *ct* "does not know" that the charge has been moved.



If the charge was moved twice, then the field lines at time *t* > *t1* would look like this—there will be two spheres, with the radiation layers between them

Three forms of synchrotron radiation



Shintake Radiation Demo Program

Radiation propagation and diffraction

- Wave propagation in free space

$$\left[\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial \boldsymbol{x}^2}+k^2\right]E_\omega(\boldsymbol{x};z)=0, \quad k=\frac{\omega}{c}=\frac{2\pi}{\lambda}$$

- Angular representation

$$\mathcal{E}_\omega(\boldsymbol{\phi};z)=\frac{1}{\lambda^2}\int d\boldsymbol{x}\; e^{-ik\boldsymbol{\phi}\cdot\boldsymbol{x}}E_\omega(\boldsymbol{x};z)$$

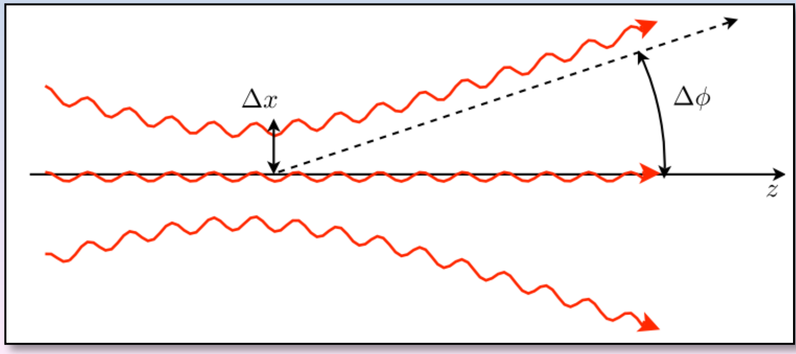
$$E_\omega(\boldsymbol{x};z)=\int d\boldsymbol{\phi}\; e^{ik\boldsymbol{\phi}\cdot\boldsymbol{x}}\mathcal{E}_\omega(\boldsymbol{\phi};z).$$

- General solution

$$E_\omega(\boldsymbol{x};z)=\int d\boldsymbol{\phi}\;\exp\left[ik(\boldsymbol{\phi}\cdot\boldsymbol{x}\pm z\sqrt{1-\phi^2})\right]\mathcal{E}_\omega(\boldsymbol{\phi};0)$$

- Paraxial approximation (ϕ^2 << 1)

$$\mathcal{E}_\omega(\boldsymbol{\phi};z)=e^{ik(1-\phi^2/2)z}\mathcal{E}_\omega(\boldsymbol{\phi};0)$$



Gaussian beam and radiation emittance

- Single electron radiation can be approximated by Gaussian beam ➔ Gaussian fundamental mode at waist *z=0*

$$E(x;0) = E_0 \exp\left(-\frac{x^2}{4\sigma_r^2}\right)$$

$$\mathcal{E}(\phi;0) = \mathcal{E}_0 \exp\left(-\frac{\phi^2}{4\sigma_{r'}^2}\right)$$

$$\sigma_r \sigma_{r'} = \frac{\lambda}{4\pi} \equiv \varepsilon_r$$

- At arbitrary *z*

$$\begin{aligned} E(x;z) &= \frac{E_0}{\sqrt{1 + i\sigma_{r'} z/\sigma_r}} \exp\left[-\frac{x^2}{4\sigma_r^2(1 + i\sigma_{r'} z/\sigma_r)}\right] \\ &= \frac{E_0}{(1 + z^2/Z_R^2)^{1/4}} \exp\left[-\frac{x^2(1 - iz/Z_R)}{4\sigma_r^2(1 + z^2/Z_R^2)} - \frac{i}{2}\tan^{-1}\left(\frac{z}{Z_R}\right)\right] \end{aligned}$$

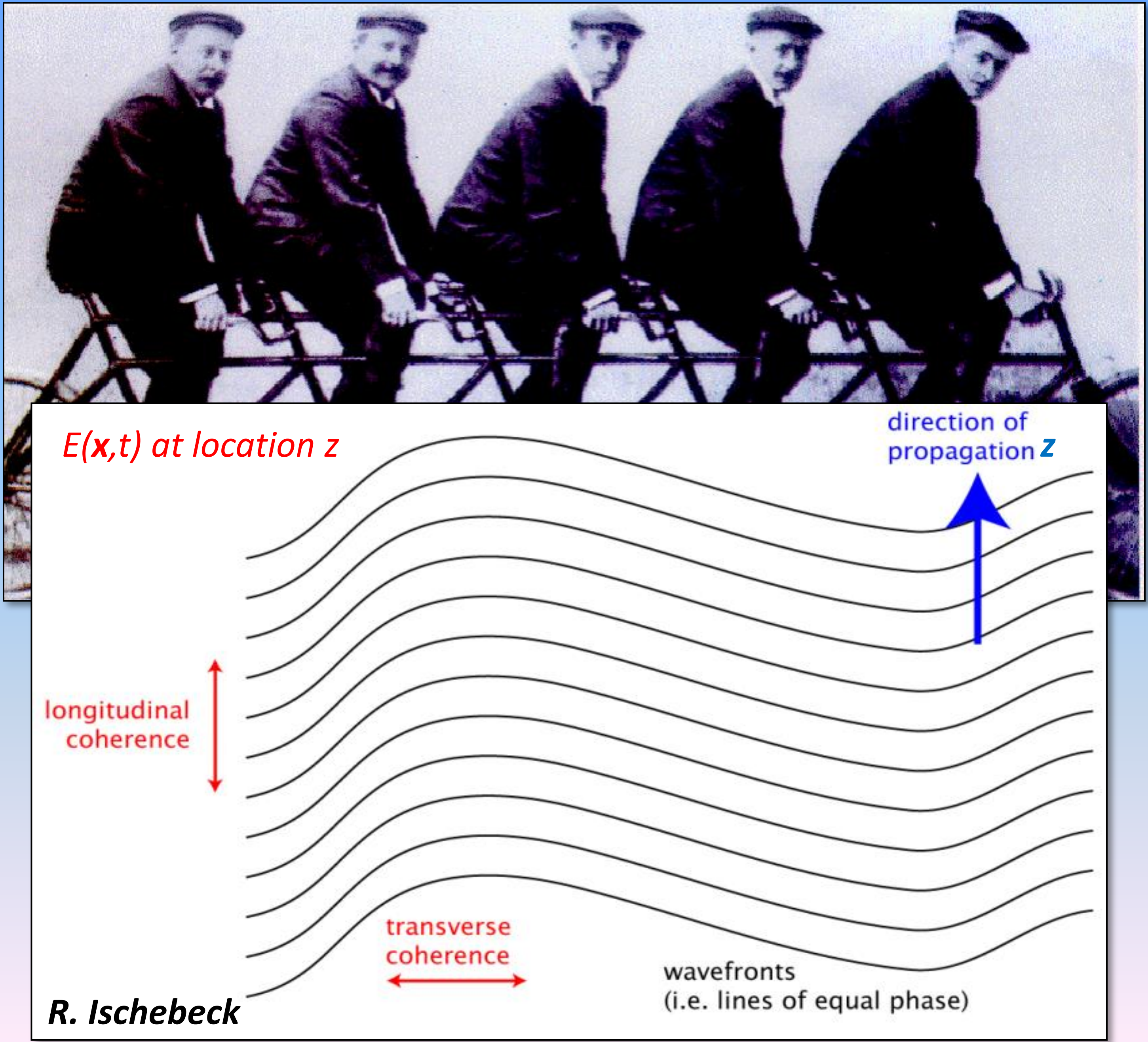
$$\sigma_r(z) = \sqrt{\frac{\lambda}{4\pi}\left(Z_R + \frac{z^2}{Z_R^2}\right)}$$

$$Z_R \equiv \sigma_r/\sigma_{r'} = 2k\sigma_r^2$$

- Analogous with electron beam

$$\varepsilon_x \leftrightarrow \frac{\lambda}{4\pi} \qquad \beta_x^* \leftrightarrow Z_R.$$

Coherence

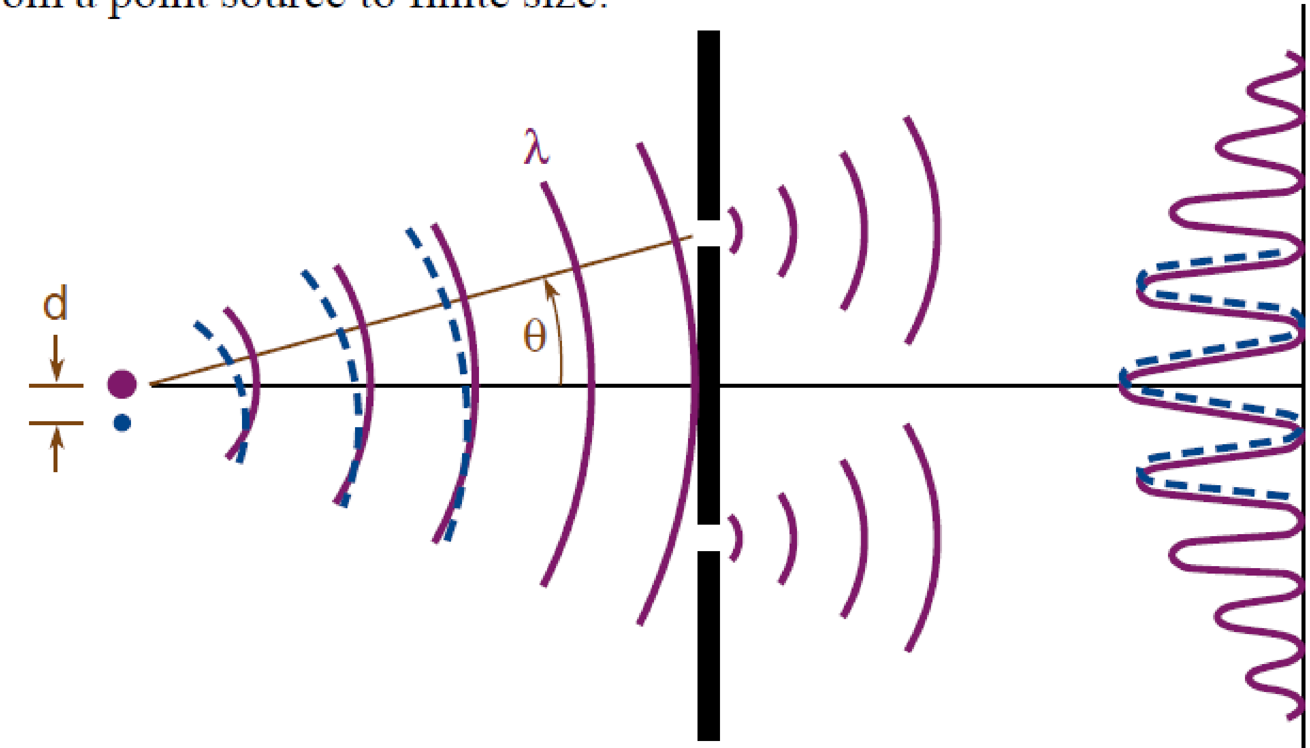


R. Ischebeck

Transverse (Spatial) Coherence

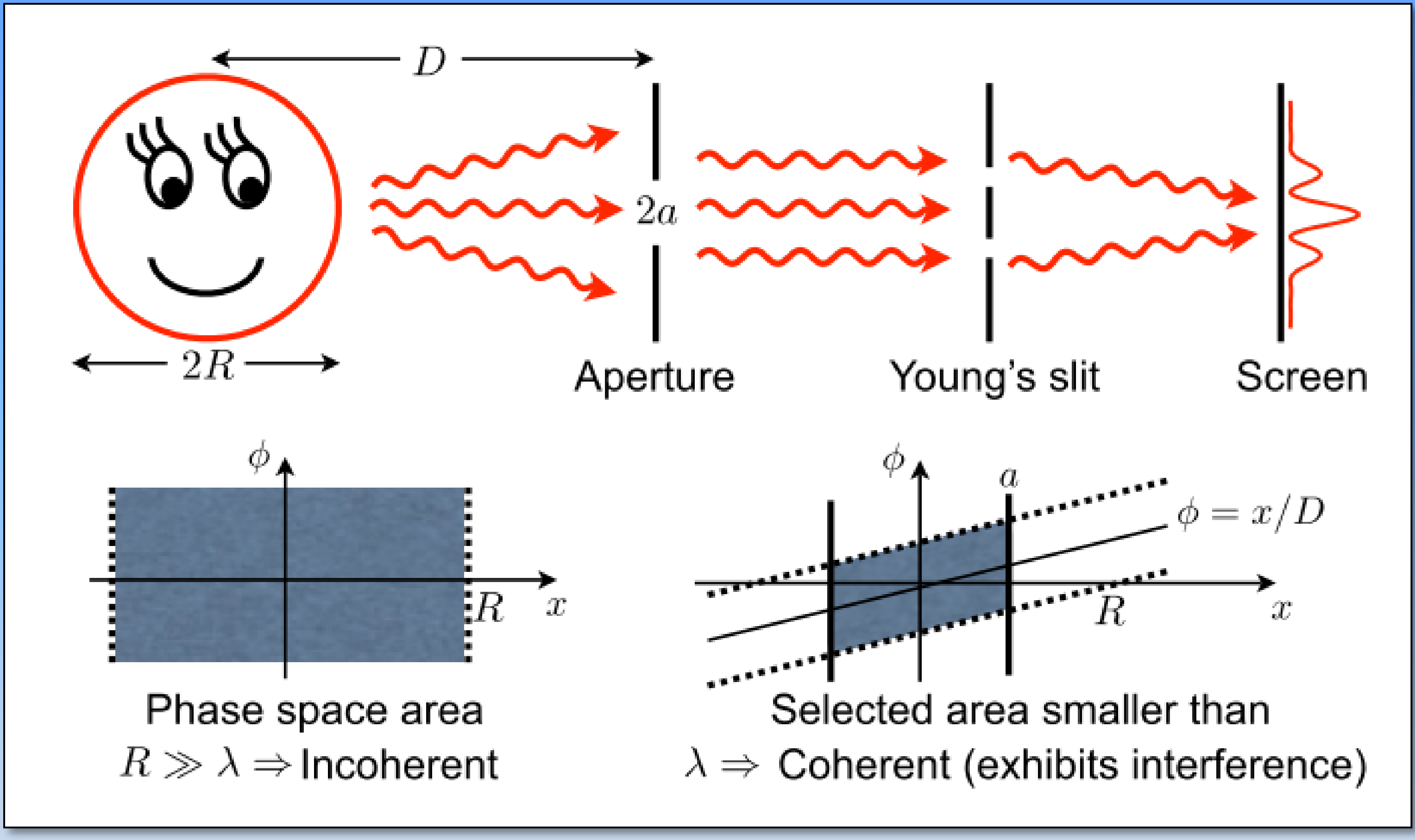
- *Transverse coherence can be measured via the interference pattern in Young's double slit experiment.*
- *Near the center of screen, transverse coherence determines fringe visibility*

Persistence of fringes as the source grows from a point source to finite size.



$$d \cdot 2\theta|_{\mathrm{FWHM}} \simeq \lambda/2$$

Phase space criteria for transverse coherence

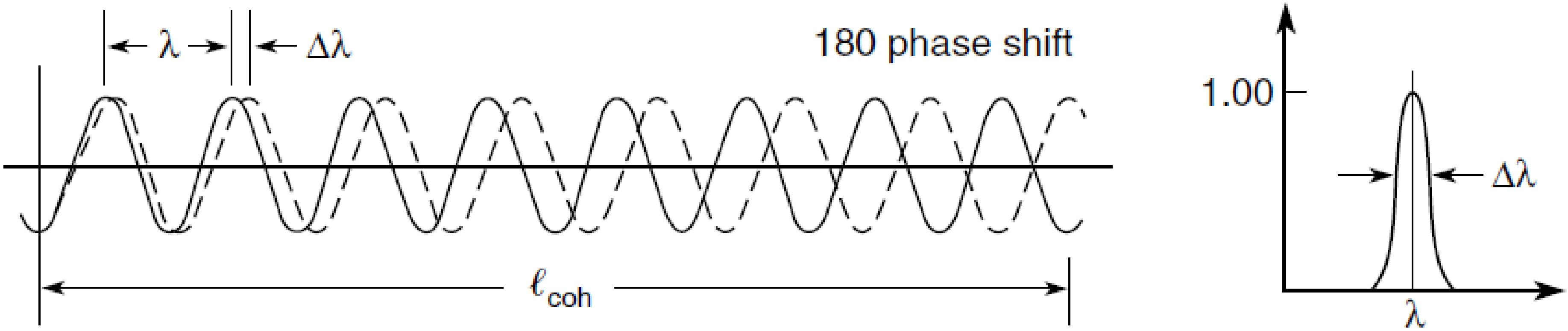


Phase space area
$R \gg \lambda \Rightarrow$ Incoherent

Selected area smaller than
$\lambda \Rightarrow$ Coherent (exhibits interference)

- Initial phase space area $4\pi R >> \lambda$
- Final phase space area $\boxed{4Ra/D \lesssim \lambda/2}$
- Coherent flux is reduced by M_T
- Show this criteria from physical optics argument

Temporal Coherence



Define a coherence length ℓ_{coh} as the distance of propagation over which radiation of spectral width Δλ becomes 180° out of phase. For a wavelength λ propagating through N cycles

$$\ell_{coh} = N\lambda$$

and for a wavelength $\lambda + \Delta\lambda$, a half cycle less $(N - \frac{1}{2})$

$$\ell_{coh} = (N - \tfrac{1}{2})\,(\lambda + \Delta\lambda)$$

Equating the two

$$N = \lambda/2\Delta\lambda$$

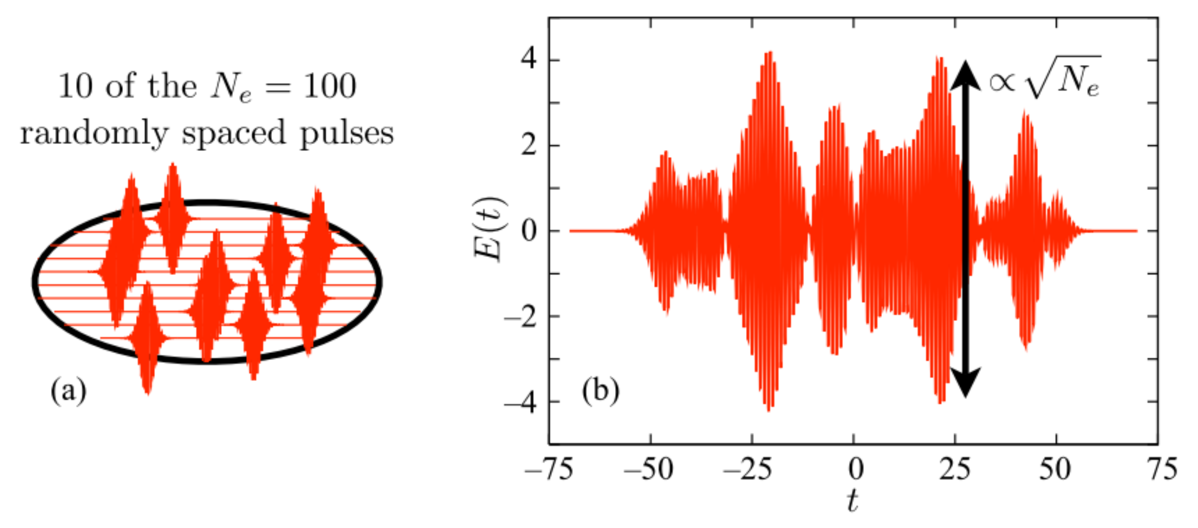
so that

$$\boxed{\ell_{coh} = \frac{\lambda^2}{2\,\Delta\lambda}}$$

Chaotic light

- Radiation from many random emitters (Sun, SR, SASE FEL)

$$E(t) = \sum_{j=1}^{N_e} E_0(t - t_j) = e_0 \sum_{j=1}^{N_e} \exp\left[-\frac{(t-t_j)^2}{4\sigma_\tau^2} - i\omega_1(t - t_j)\right]$$



- Correlation function and coherence time

$$\mathcal{C}(\tau) \equiv \frac{\left\langle \int dt\ E(t)E^*(t+\tau)\right\rangle}{\left\langle \int dt\ |E(t)|^2\right\rangle}$$

$$t_{\mathrm{coh}} \equiv \int dt\ |\mathcal{C}(\tau)|^2$$

Temporal mode and fluctuation

- Number of regular temporal regions is # of coherent modes

$$M_L \approx \frac{T}{t_{\rm coh}} = \frac{T}{2\sqrt{2\pi}\sigma_\tau} \approx \frac{T}{5\sigma_\tau}.$$

- Intensity fluctuation $\frac{\Delta W}{W} = \frac{1}{\sqrt{M_L}}$

- Same numbers of mode in frequency domain

$$E_\omega = \frac{e_0\sigma_\tau}{\sqrt{\pi}} \sum_{j=1}^{N_e} \exp\left[-\frac{(\omega-\omega_1)^2}{4\sigma_\omega^2} + i\omega t_j\right]$$

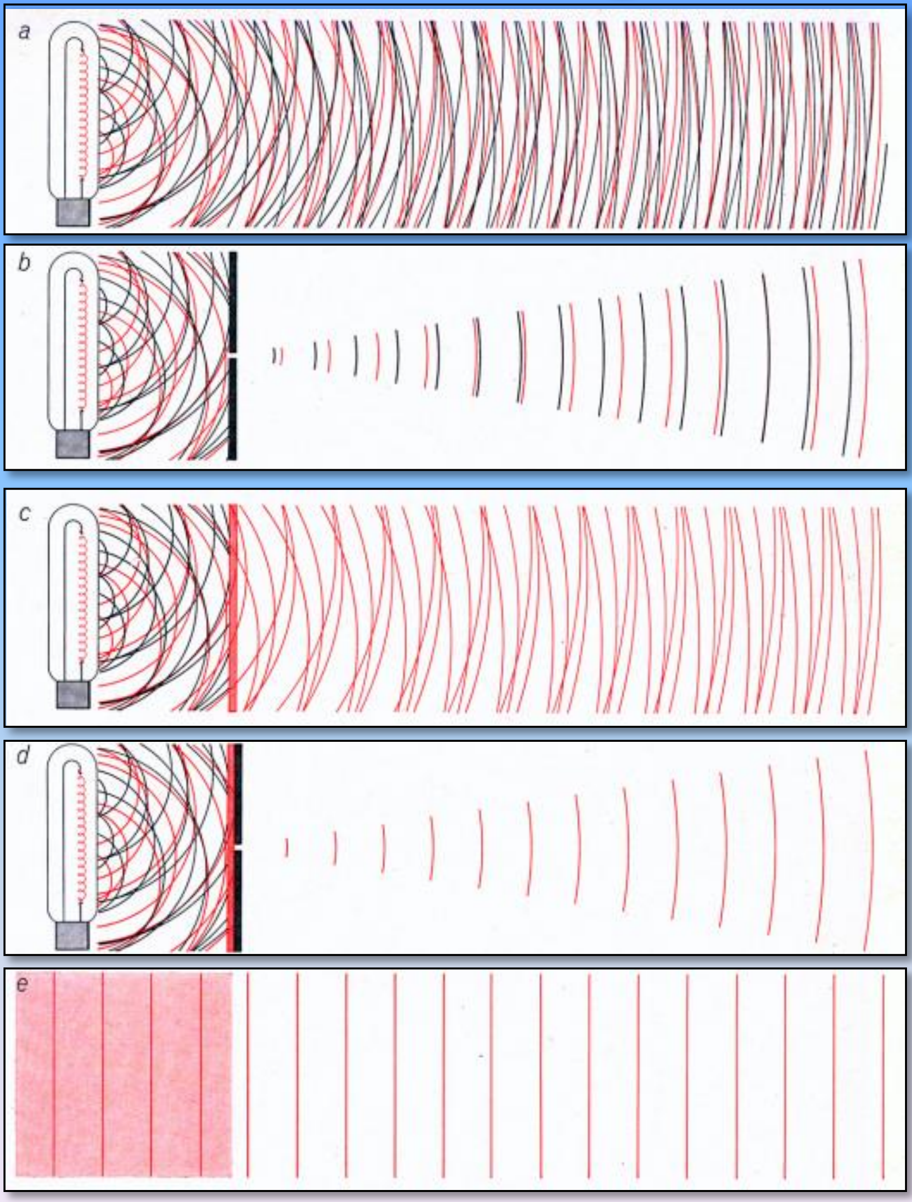
$$c\sigma_\tau \cdot \frac{\sigma_\omega}{\omega_1} = \frac{\lambda_1}{4\pi},$$

Fourier limit, minimum longitudinal phase space

- Longitudinal phase space is M_L larger than Fourier limit

- Total # of modes $M = M_L M_T^2.$

Light Bulb vs. Laser



Radiation emitted from light bulb is chaotic.

Pinhole can be used to obtain spatial coherence.

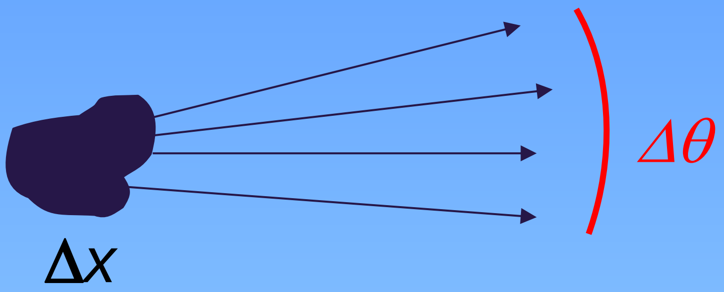
Monochromator can be used to obtain temporal coherence.

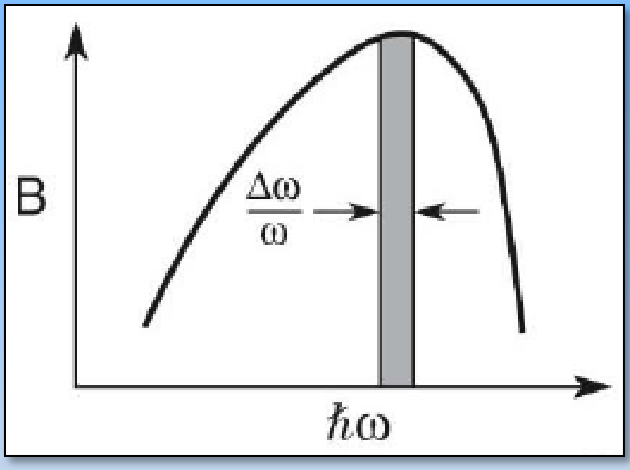
Pinhole and Monochromator can be combined for coherence.

Laser light is spatially and temporally coherent.

A. Schawlow (Nobel prize on laser spectroscopy***), Scientific Americans, 1968***

Brightness





$$B = \frac{\text{Photons in unit spectral range in unit time}}{(\text{source size} \times \text{divergence})^2}$$

→ **Peak**

Average

Units: photons/s/mm²/mrad²/0.1%BW

Incoherent radiation from many electrons

- Such a beam can be described by the convolution of the coherent Gaussian beam with the electron distribution in phase space

- Effective source size and divergence

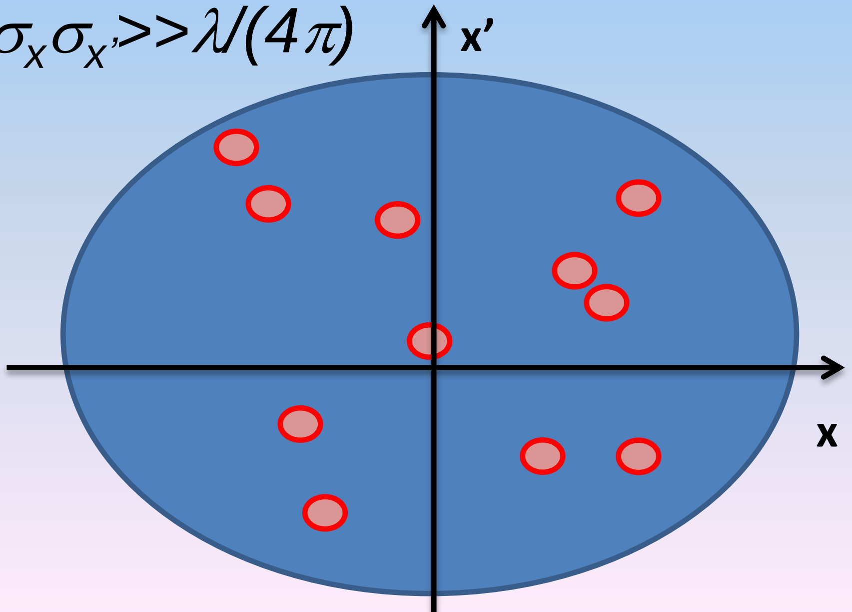
$$\Sigma_x = \sqrt{\sigma_x^2 + \sigma_r^2} \qquad \Sigma_{x'} = \sqrt{\sigma_{x'}^2 + \sigma_{r'}^2}.$$

- When electron beam emittance $\sigma_x\sigma_{x'} >> \lambda/(4\pi)$

$$\Sigma_x \Sigma_{x'} \gg \frac{\lambda}{4\pi}$$

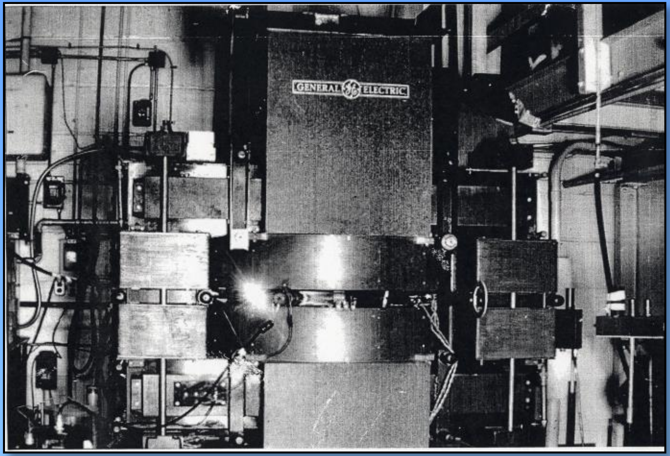
- # of transverse modes

$$M_T = \frac{\Sigma_x \Sigma_{x'}}{\lambda/4\pi} = \frac{\varepsilon_x}{\varepsilon_r}$$

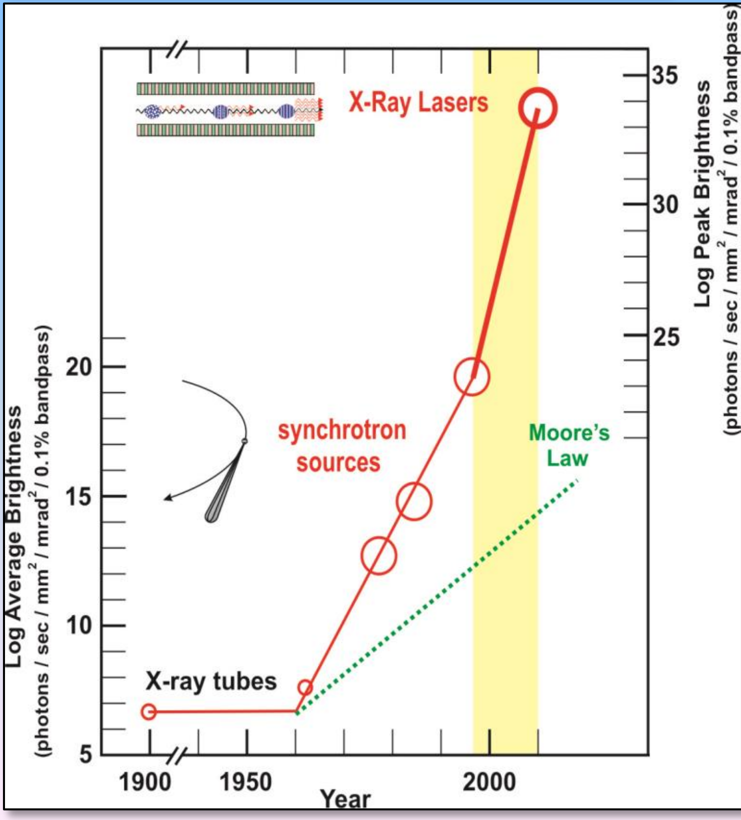


Evolution of X-ray Light Sources

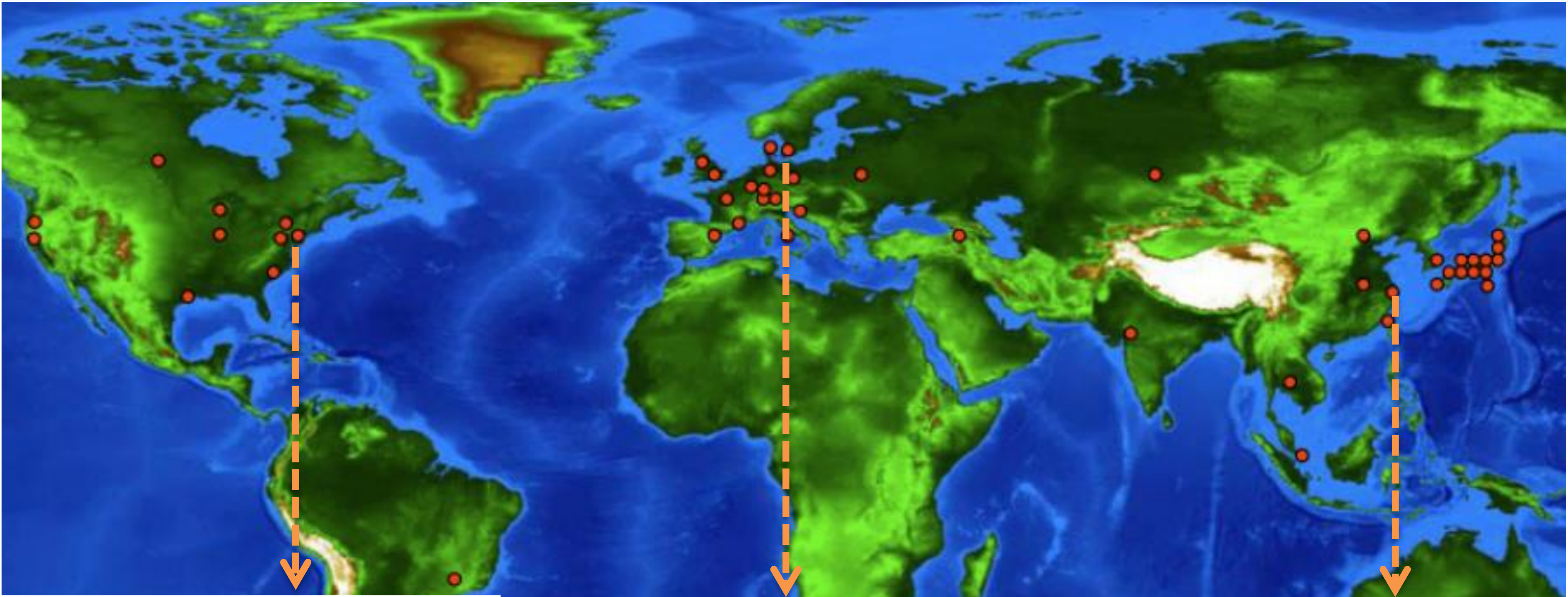
- *GE synchrotron (1946) opened a new era of accelerator-based light sources.*



- *These light sources have evolved rapidly over four generations.*
  - *The first three-generations are based on synchrotron radiation.*
  - *The forth-generation light source is a game-changer based on FELs.*
- *The dramatic improvement of brightness and coherence over 60 years easily outran Moore's law.*



Synchrotron Radiation Facilities



NSLS-II (2015)

MAX-IV (2016)

SSRF (2009)

- *State-of-art storage rings have **pulse duration ~10 ps, emittance ~1 nm.***
- *Diffraction-limited storage rings with **emittance ~10 pm** are under active R&D.*

Radiation intensity

- What if emitters are not random in time

$$\left\langle \left|E(\omega)\right|^2 \right\rangle = \left|E_\omega^0\right|^2 \left\langle \left| \sum_{j=1}^{N_e} e^{i\omega t_j} \right|^2 \right\rangle$$

$$\left\langle \left| \sum_{j=1}^{N_e} e^{i\omega t_j} \right|^2 \right\rangle = N_e + \left\langle \sum_{j \neq k}^{N_e} e^{i\omega(t_j - t_k)} \right\rangle$$

$$\left\langle \left| \sum_{j \neq k}^{N_e} e^{i\omega(t_j - t_k)} \right|^2 \right\rangle = N_e(N_e - 1) \left| \int dt\ f(t) e^{i\omega t} \right|^2$$

- For an electron bunch with rms bunch length σ_e

$$f(t) = \frac{1}{\sqrt{2\pi}\,\sigma_e} \exp\left(-\frac{t^2}{2\sigma_e^2}\right)$$

$$\left\langle \left|E(\omega)\right|^2 \right\rangle = N_e \left|E_\omega^0\right|^2 \left[1 + (N_e - 1) e^{-\omega^2 \sigma_e^2}\right]$$

- When $(N_e - 1) e^{-\omega^2 \sigma_e^2} \ll 1$

  intensity from many electrons add **incoherently (~*N*$_e$)**

Bunching and coherent radiation

- If the bunch length is shorter than the radiation wavelength

$$(N_e - 1)e^{-\omega^2\sigma_e^2} \geq 1$$

$$\left\langle |E(\omega)|^2 \right\rangle = N_e |E_\omega^0|^2 \left(1 + (N_e - 1)\,\underset{\uparrow}{|f(\omega)|^2}\right)$$

Form factor or bunching factor

- Radiation intensity from many electrons add **coherently (~N_e^2)**
- Another way to produce bunching from a relatively long bunch is through so-called microbunching