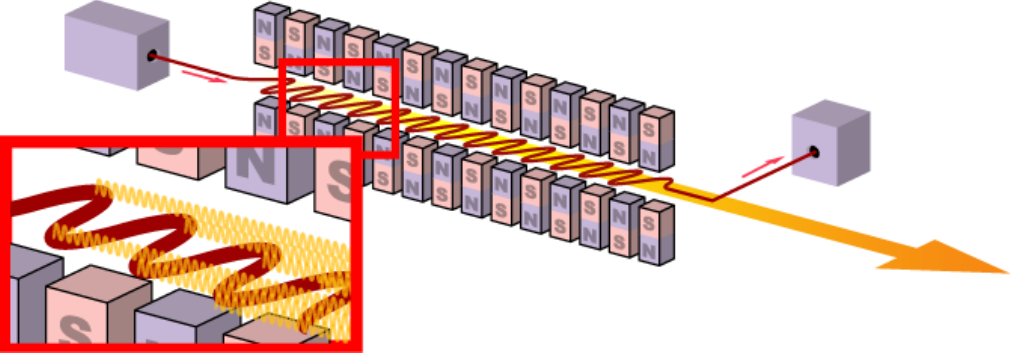
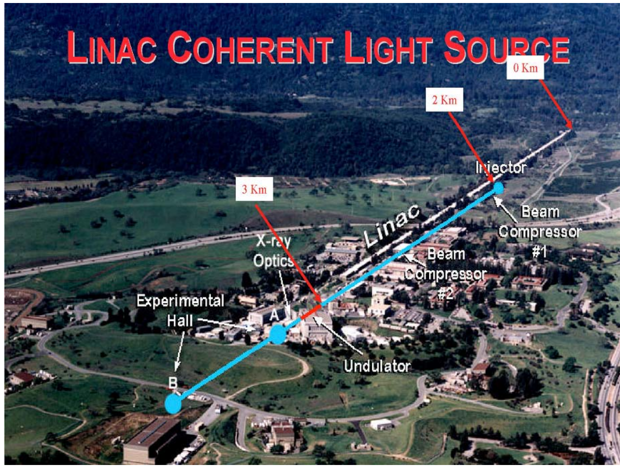


Introduction to the Physics of Free-Electron Lasers



Outline

FEL Theory:

1-D FEL Equations

FEL Instability

Universal Scaling

SASE vs Seeded

Non-ideal effects

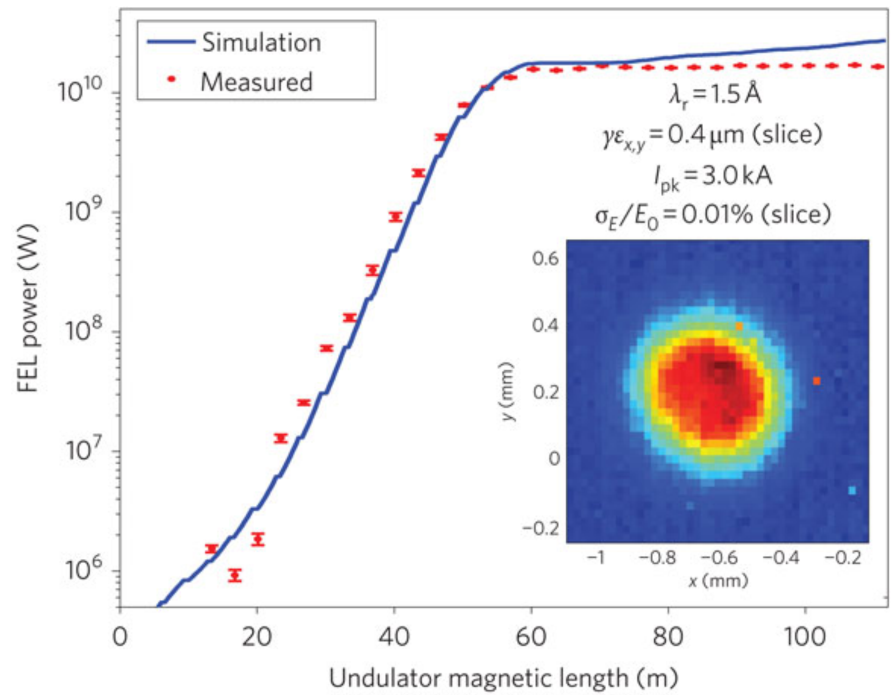
The X-Ray Free-Electron Laser

SLAC

For Angstrom level radiation:
High energy linac (~5-15 GeV)
+
Long Undulator (120 m)

X-FEL shares properties of conventional lasers:

- -High Power (~ 10-100 GW)
- -Short Pulse (~4-100 fs)
- -Narrow Bandwidth (~0.1% to 0.005%)
- -Transverse Coherence



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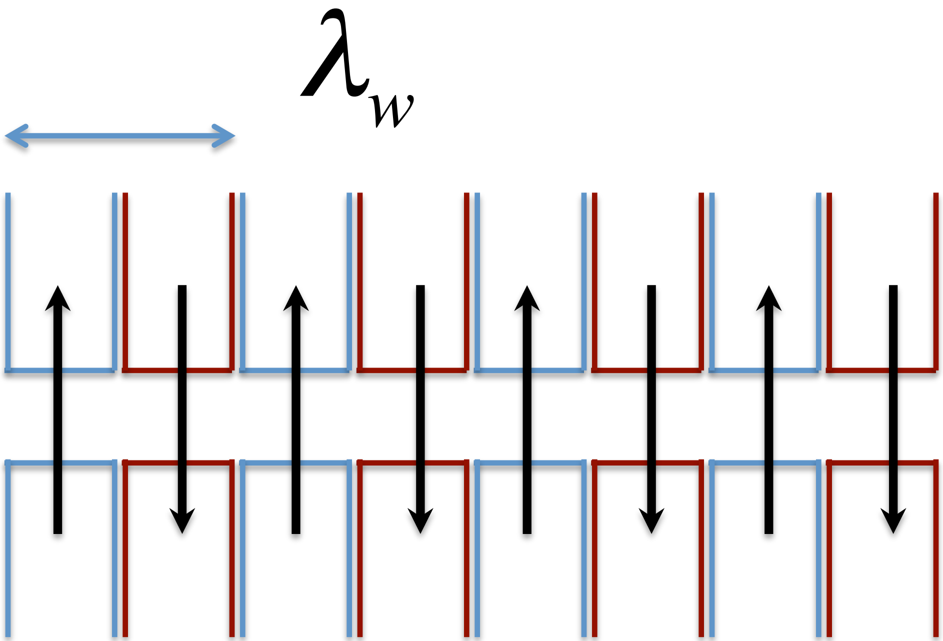
First lasing and operation of an ångstrom-wavelength free-electron laser

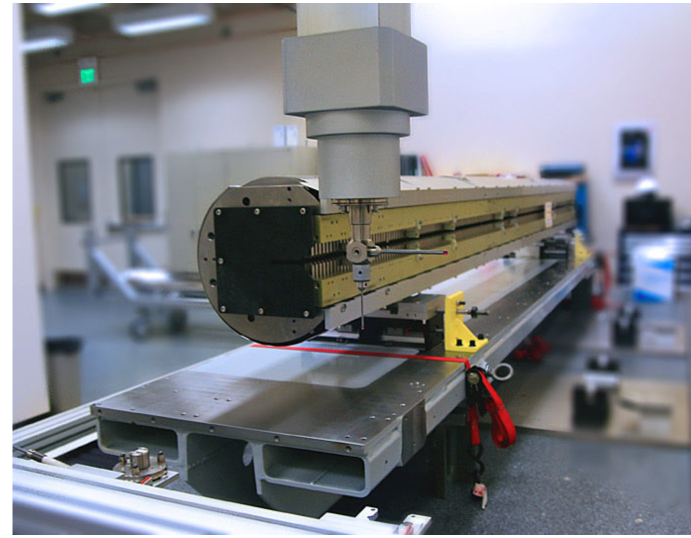
P. Emma[1]*, R. Akre[1], J. Arthur[1], R. Bionta[2], C. Bostedt[1], J. Bozek[1], A. Brachmann[1], P. Bucksbaum[1], R. Coffee[1], F.-J. Decker[1], Y. Ding[1], D. Dowell[1], S. Edstrom[1], A. Fisher[1], J. Frisch[1], S. Gilevich[1], J. Hastings[1], G. Hays[1], Ph. Hering[1], Z. Huang[1], R. Iverson[1], H. Loos[1], M. Messerschmidt[1], A. Miahnahri[1], S. Moeller[1], H.-D. Nuhn[1], G. Pile[3], D. Ratner[1], J. Rzepiela[1], D. Schultz[1], T. Smith[1], P. Stefan[1], H. Tompkins[1], J. Turner[1], J. Welch[1], W. White[1], J. Wu[1], G. Yocky[1] and J. Galayda[1]

10 Billion times brighter than Synchrotron Radiation Sources!!!!

Undulator

Periodic array of dipole magnets with alternating polarity





$$B_y = B_0 \sin(k_w z)$$

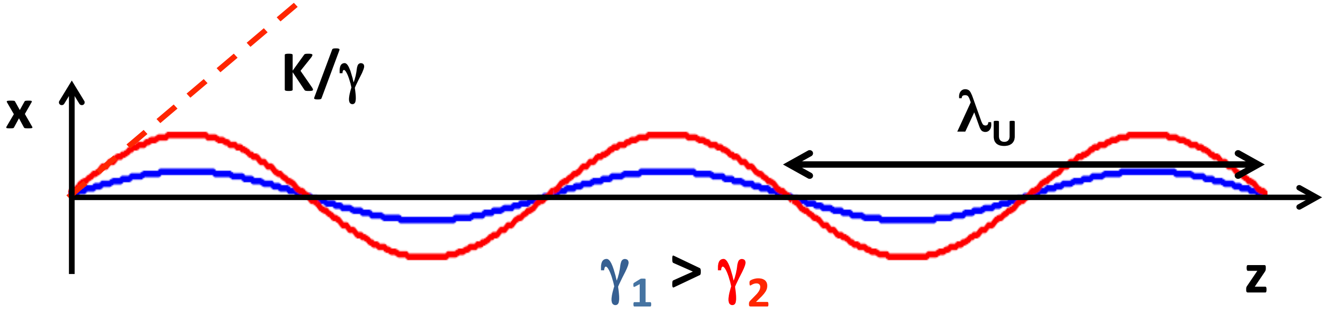
$$k_w = 2\pi / \lambda_w$$

Electron Motion in Undulator

$$\vec{P} = mc\vec{\beta}\gamma + e\vec{A} = const$$

$$\vec{A} = \hat{x}\frac{B_0}{ek_w mc}\cos(k_w z) = -\hat{x}K\cos(k_w z) \qquad K = \left|\frac{B_0}{ek_w mc}\right|$$

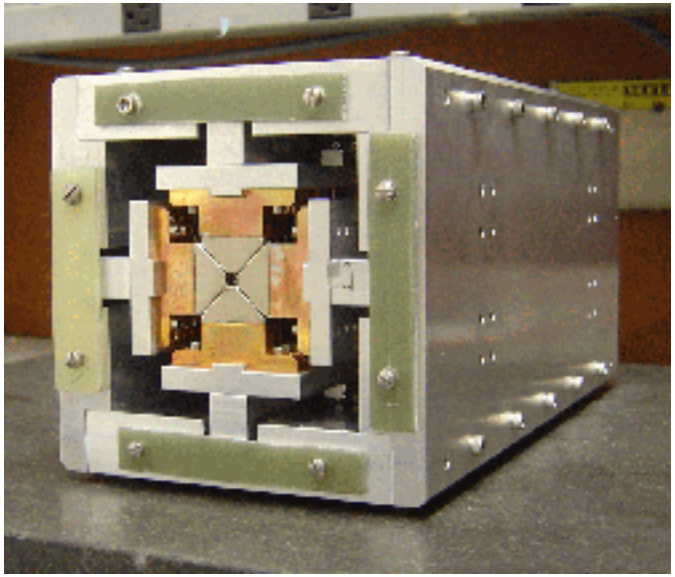
$$\vec{\beta} = \hat{x}\frac{K}{\gamma}\cos(k_w z)$$



Helical Undulator

$$B_x = -B_0 \sin(k_U z)$$

$$B_y = B_0 \cos(k_U z)$$

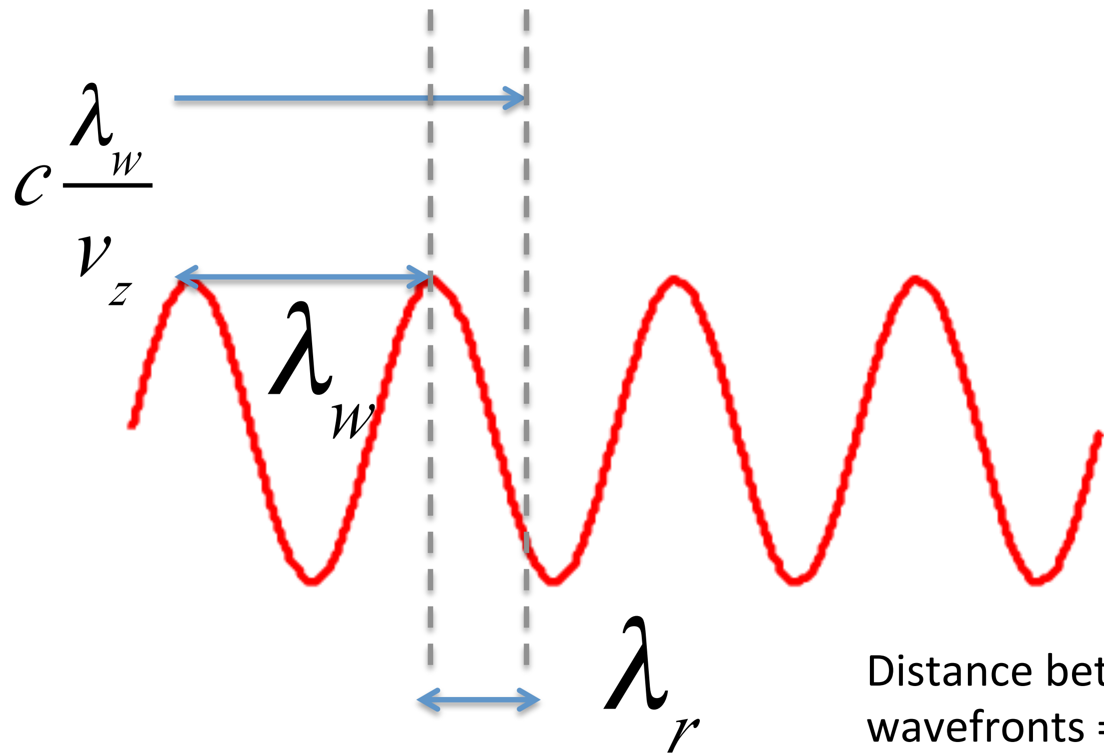


$$\beta_x(z) = -(K/\gamma)\sin(k_U z)$$

$$\beta_y(z) = (K/\gamma)\cos(k_U z)$$

Trajectory is a helix

Undulator Radiation

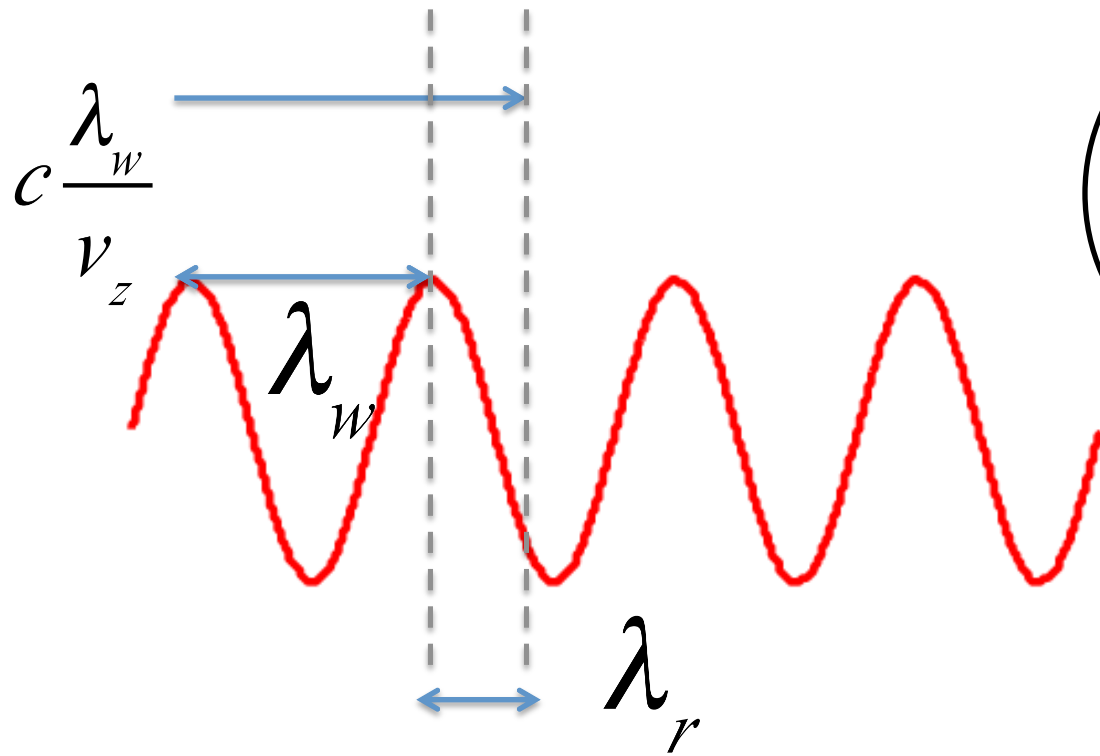


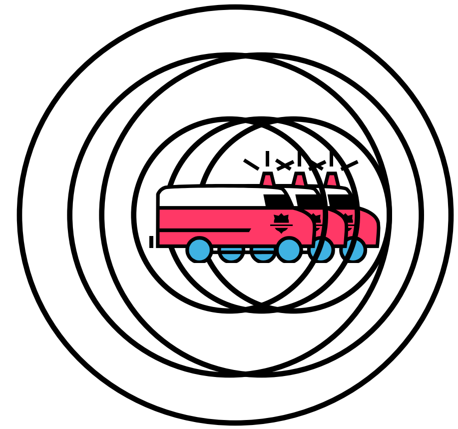
Distance between two consecutive wavefronts = wavelength

Note: light slips ahead by 1 wavelength per oscillation period

$$\lambda_r = \frac{\lambda_w}{\beta_z} - \lambda_w$$

Undulator Radiation





Dopppler shift of undulator period!

$$\lambda_r = \frac{\lambda_w}{\beta_z} - \lambda_w$$

Distance between two consecutive wavefronts = wavelength

Note: light slips ahead by 1 wavelength per oscillation period

Central Wavelength

$$\lambda_r = \frac{\lambda_w}{\beta_z} - \lambda_w$$

$$\frac{1}{\gamma^2} = 1 - \beta_z^2 - \vec{\beta}_\perp^2$$

$$\frac{v_z}{c} = \sqrt{1 - \frac{1}{\gamma^2} - \vec{\beta}_\perp^2}$$

Central Wavelength

$$\lambda_r = c\frac{\lambda_w}{v_z} - \lambda_w$$

$$\frac{1}{\gamma^2} = 1 - \beta_z^2 - \vec{\beta}_\perp^2$$

<<1

$$\frac{v_z}{c} = \sqrt{1 - \frac{1}{\gamma^2} - \vec{\beta}_\perp^2} \approx 1 - \frac{1}{2\gamma^2} - \frac{\vec{\beta}_\perp^2}{2}$$

Central Wavelength

$$\frac{v_z}{c} \approx 1 - \frac{1}{2\gamma^2} - \frac{\vec{\beta}_\perp^2}{2}$$

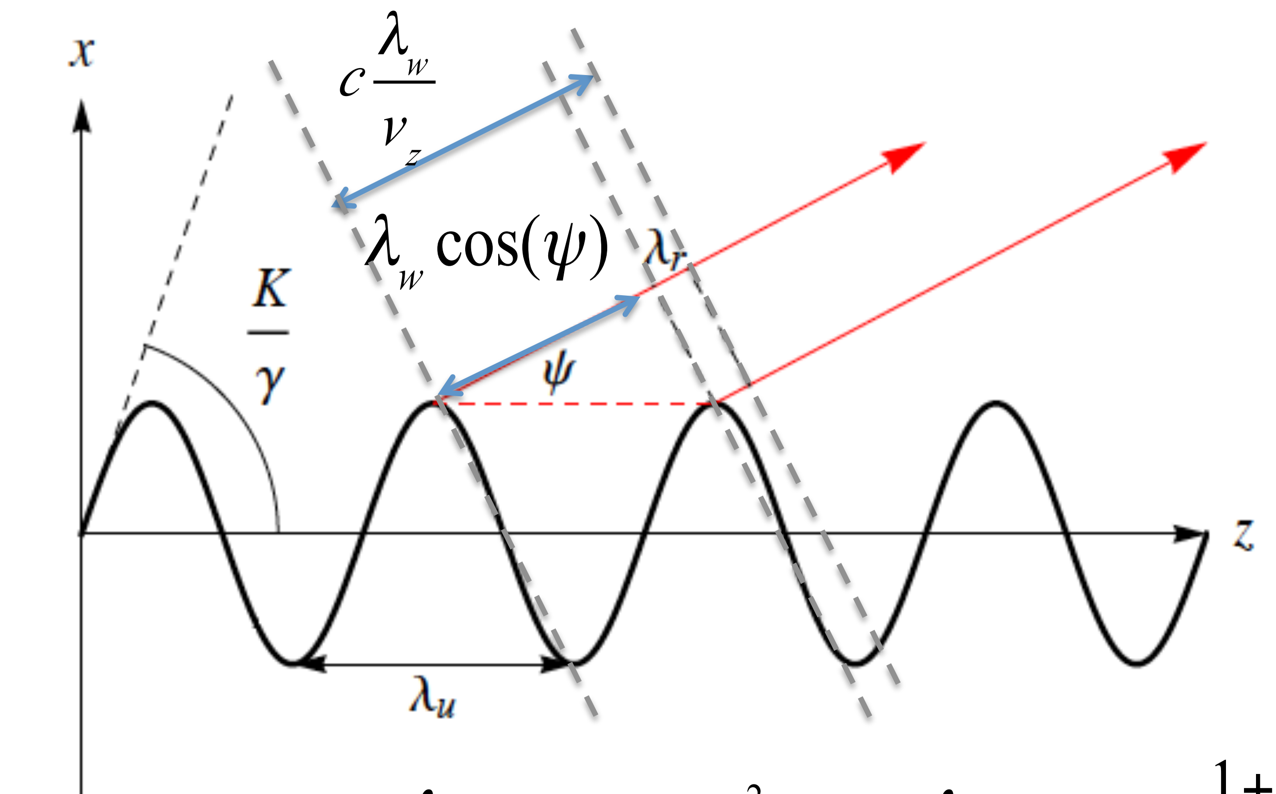
$$\lambda_r = c\frac{\lambda_w}{v_z} - \lambda_w$$

$$\frac{c}{v_z} \approx 1 + \frac{1}{2\gamma^2} + \frac{\vec{\beta}_\perp^2}{2}$$

Helical undulator $\vec{\beta}_\perp^2 = \frac{K^2}{\gamma^2} << 1 \longrightarrow \lambda_r \approx \lambda_w \frac{1+K^2}{2\gamma^2}$

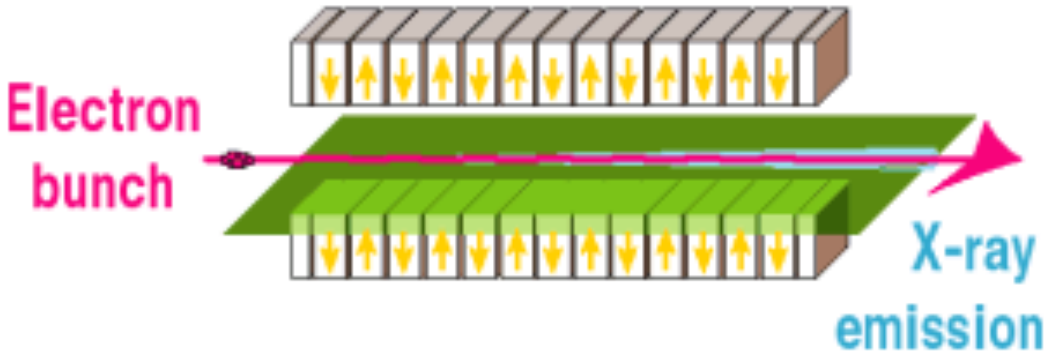
Helical undulator $\vec{\beta}_\perp^2 = \frac{K^2}{\gamma^2}\cos^2(k_w z) \longrightarrow \lambda_r \approx \lambda_w \frac{1+\frac{K^2}{2}}{2\gamma^2}$

Central Wavelength



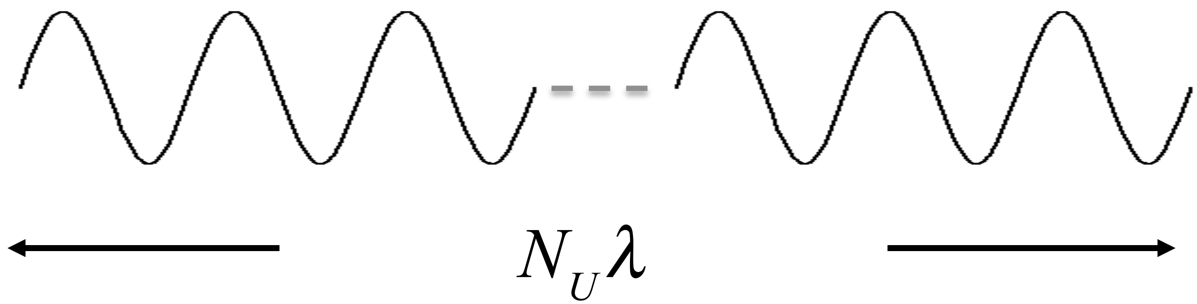
$$\lambda_r = \lambda_w \cos(\psi) - \frac{\lambda_w}{\beta_z} \approx \lambda_w \left(1 - \frac{\psi^2}{2}\right) - \frac{\lambda_w}{\left(\frac{1 - \frac{K^2}{2}}{2\gamma^2}\right)} \approx \lambda_w \frac{1 + \frac{K^2}{2} + \gamma^2 \psi^2}{2\gamma^2}$$

Undulator radiation, single electron



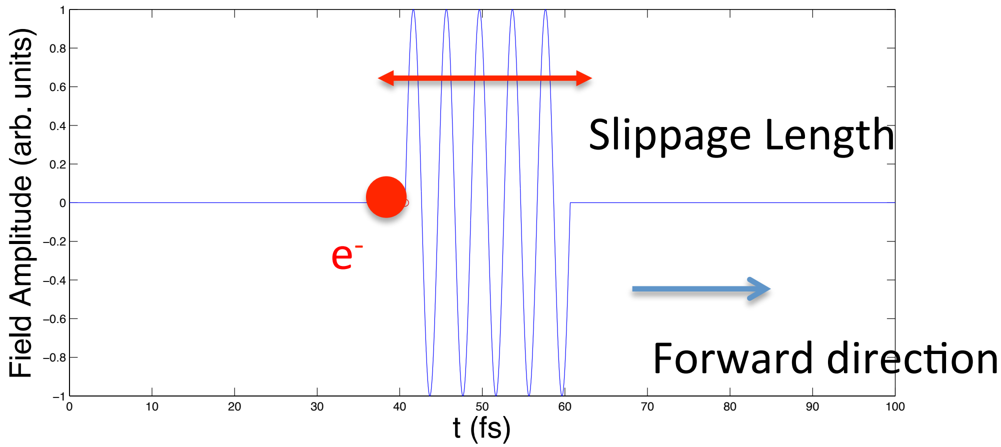
Undulator with N_U periods.

$$\Delta\lambda / \lambda \simeq 1 / N_U$$



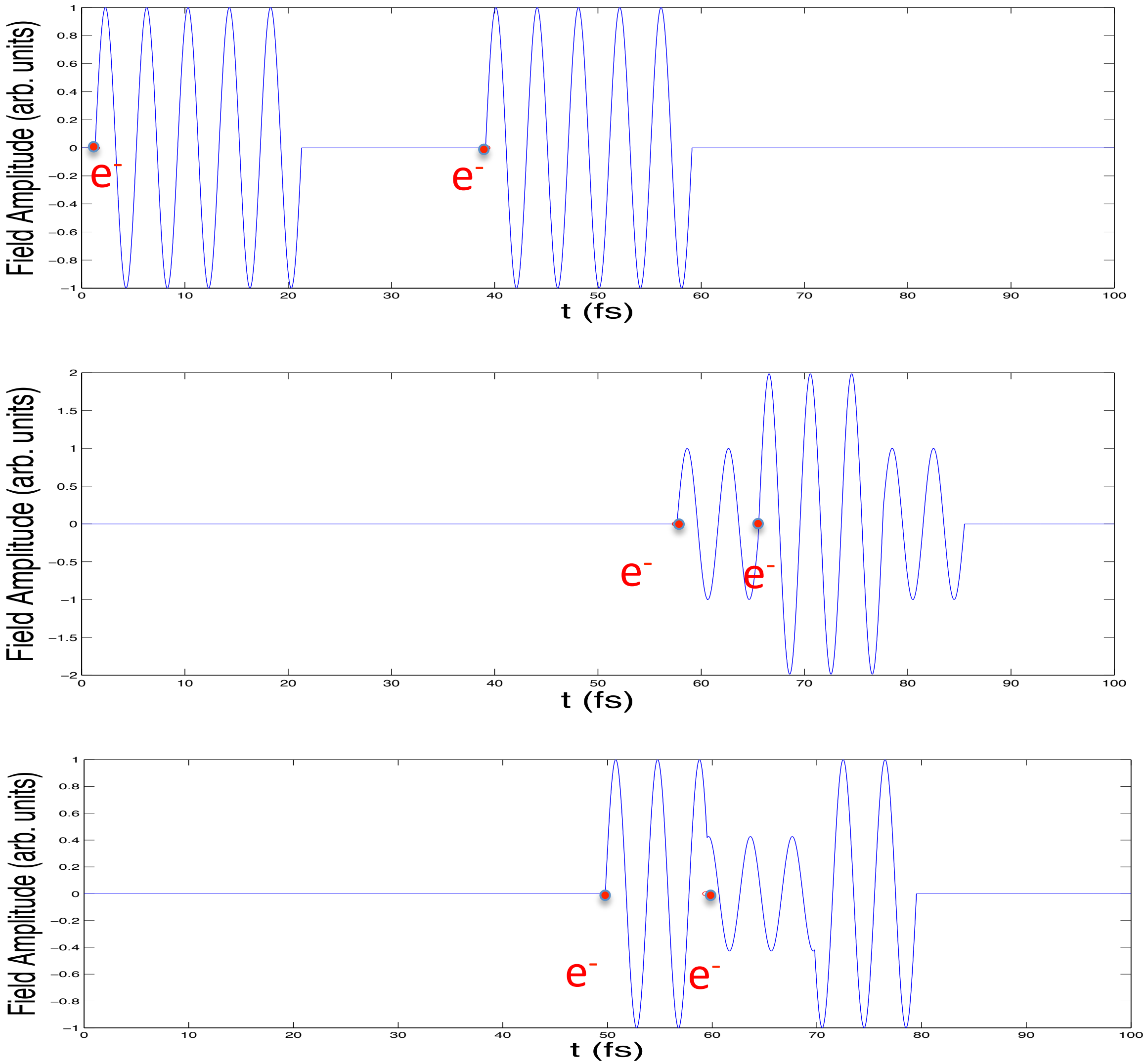
Each electron emits a wave train with N_U cycles
In the forward direction
Polarization: linear for linear undulator
circular for helical undulator

Incoherent Undulator Radiation

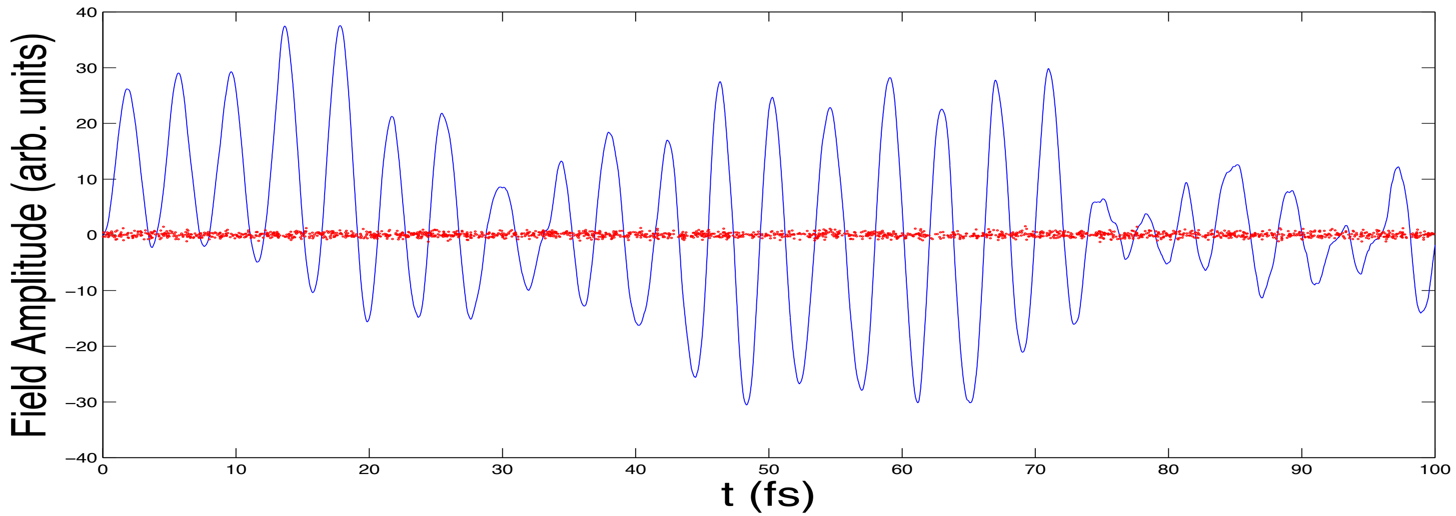


1 particle -> 1 wavetrain

Incoherent Undulator Radiation



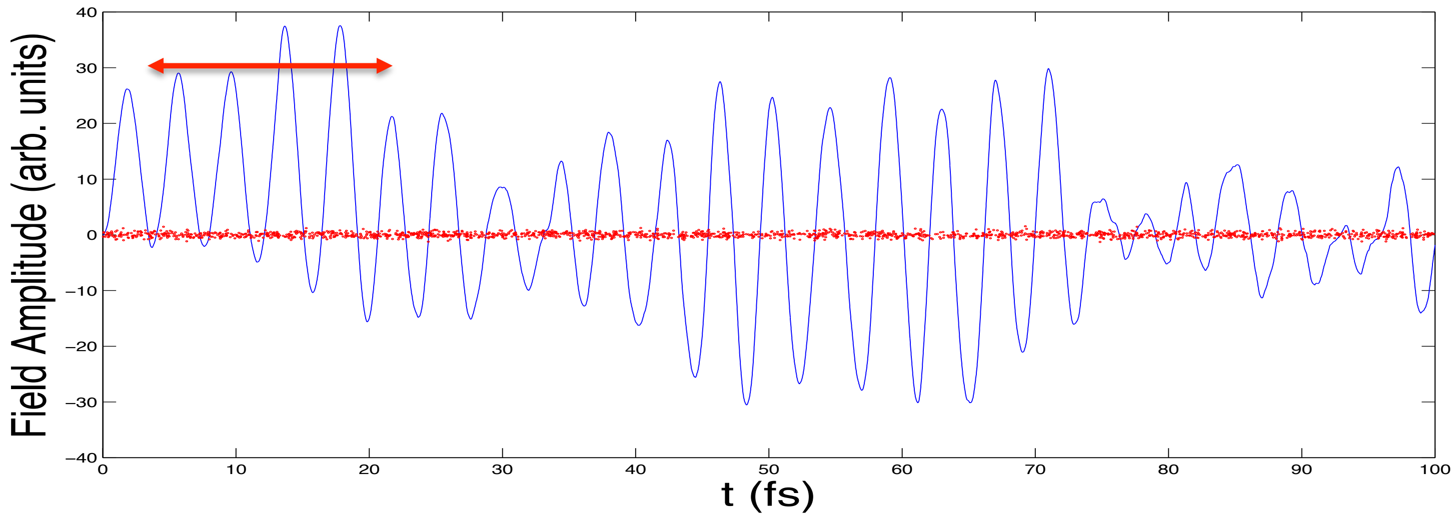
Incoherent Undulator Radiation



Electrons randomly distributed: power $\propto$ number of particles **in a wavetrain**

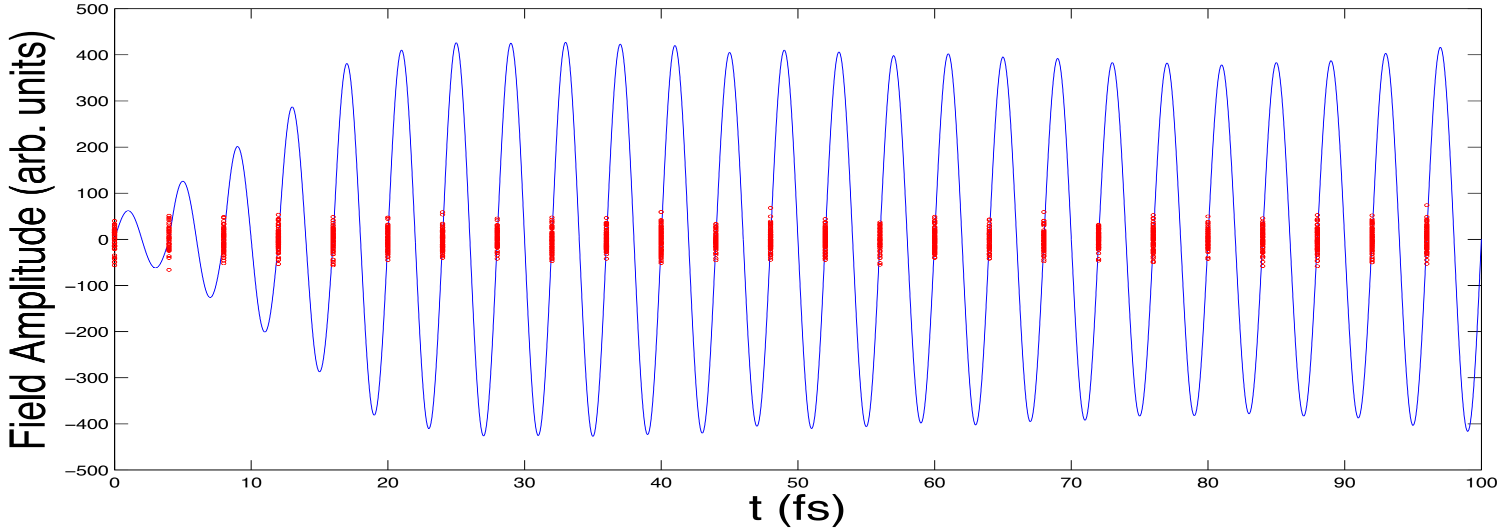
$$<\left(\sum_n E_0 \sin(\omega t+\theta_n)\right)^2>=<\sum_n \left(E_0 \sin(\omega t+\theta_n)\right)^2>+<\sum_{n,m} E_0^2 \sin(\omega t+\theta_n)\sin(\omega t+\theta_m)>=\frac{NE_0^2}{2}$$

Incoherent Undulator Radiation



Coherence Length = $N_u \lambda$ (length of a wavetrain)

Coherent Emission



Particles are bunched at multiples of the wavelength

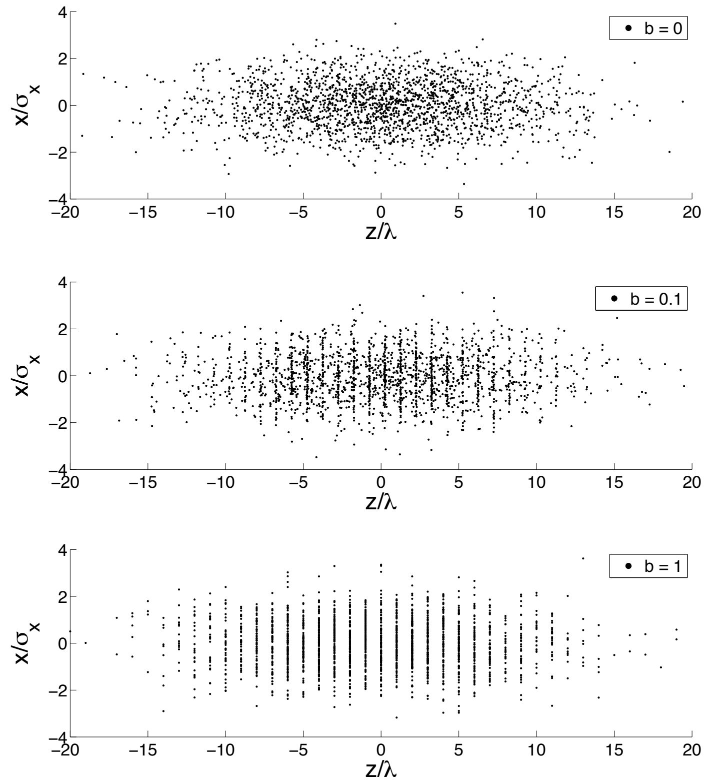
power $\propto$ square of number of particles **in a wavetrain**

$$< \left(\sum_n E_0 \sin(\omega t + \theta_0) \right)^2 >= \frac{N^2 E_0^2}{2}$$

For typical x-ray FELs N ~10^7

Huge gain going from incoherent to coherent emission!

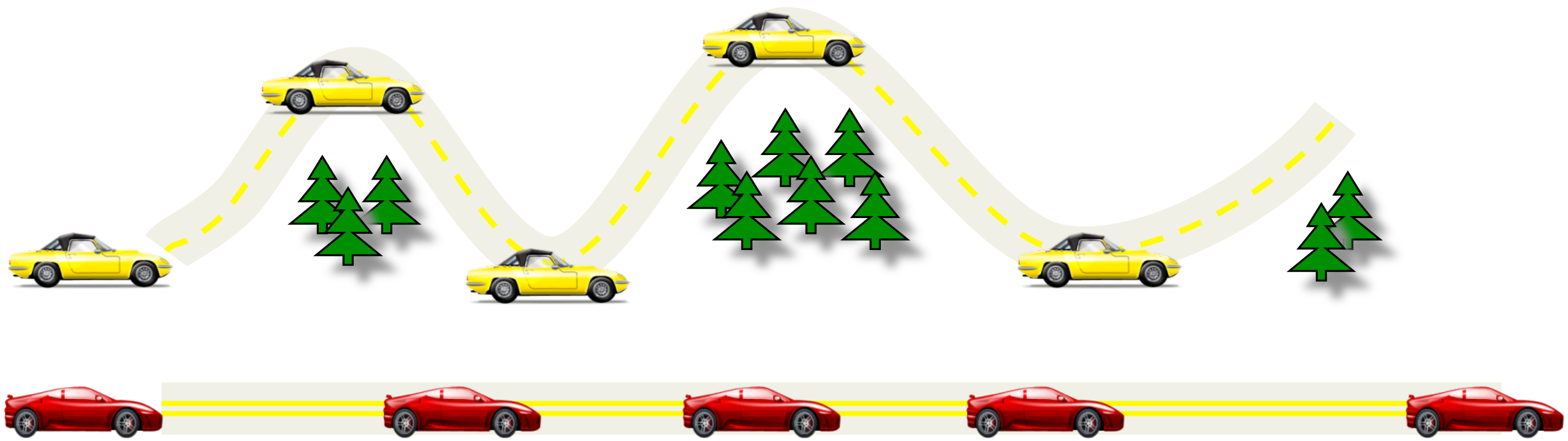
FEL: Working Principle



$$\frac{dP}{d\Omega d\omega} = \frac{dP_{sp}}{d\Omega d\omega} F(\vec{k}_{\perp}) N^2 \, |b(k)|^2$$

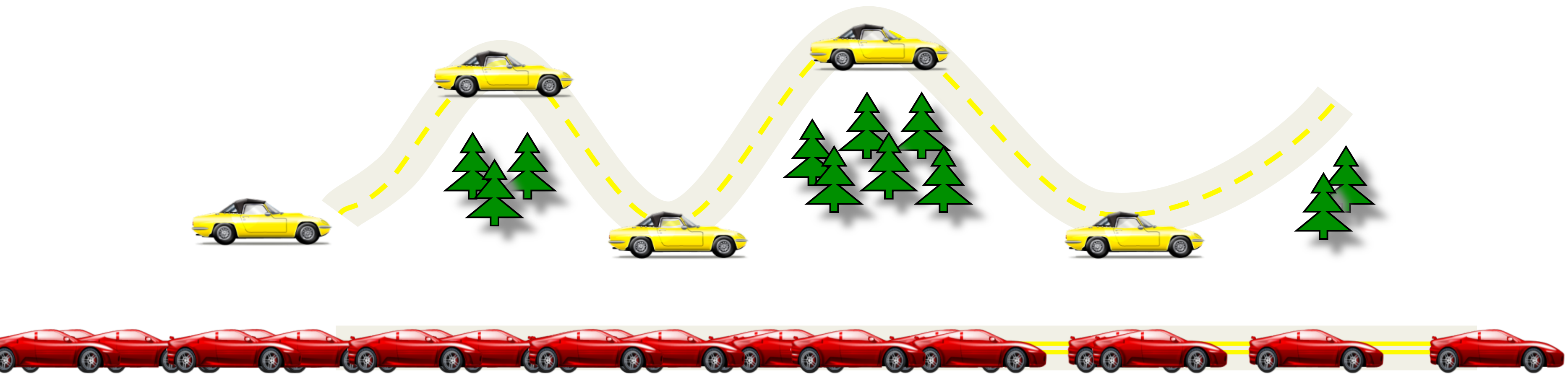
$$b(k) = \frac{1}{N} \sum_n e^{-ikz_n}$$

Resonant Interaction



Courtesy of D. Ratner

The Resonance Condition



Courtesy of D. Ratner

Working Principle

Resonant Interaction

↓

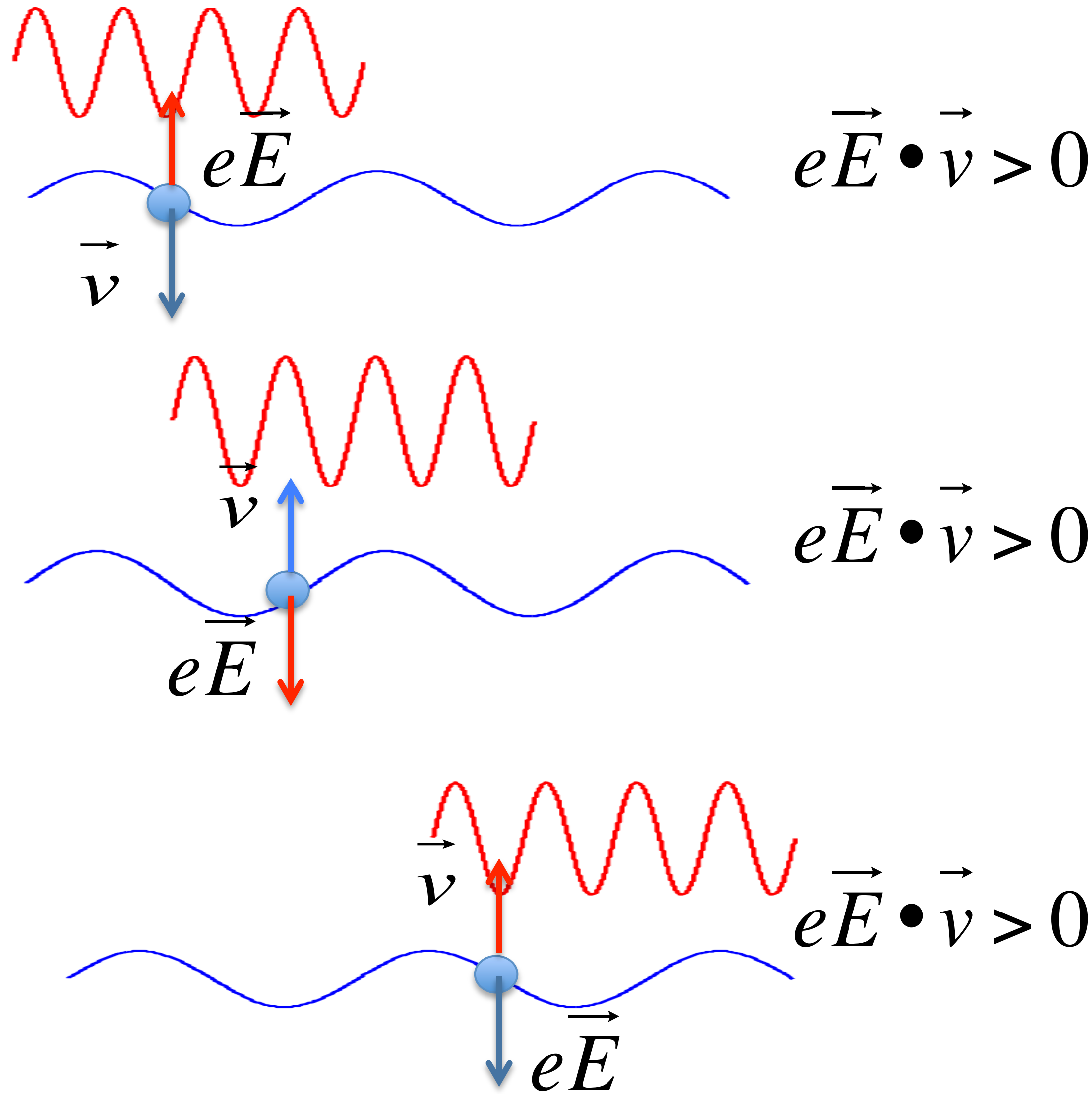
Energy Modulation

↓

Density Modulation

↓

Coherent Radiation



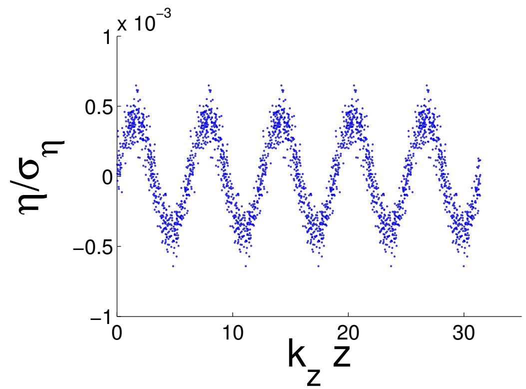
Working Principle

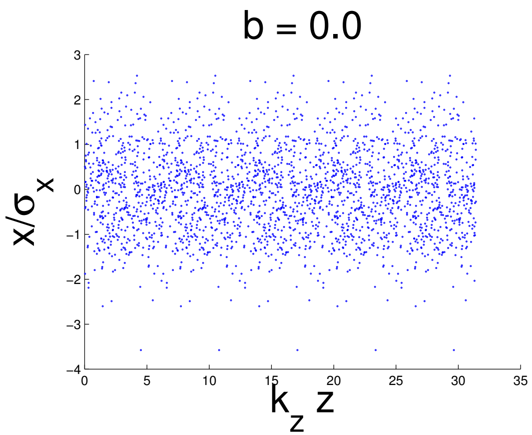
Resonant Interaction

Energy Modulation

Density Modulation

Coherent Radiation





Working Principle

Resonant Interaction

↓

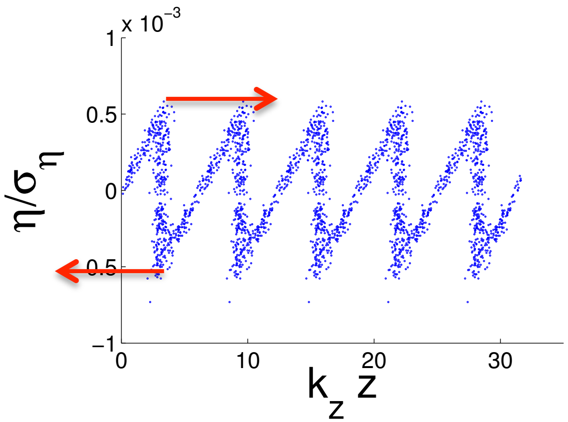
Energy Modulation

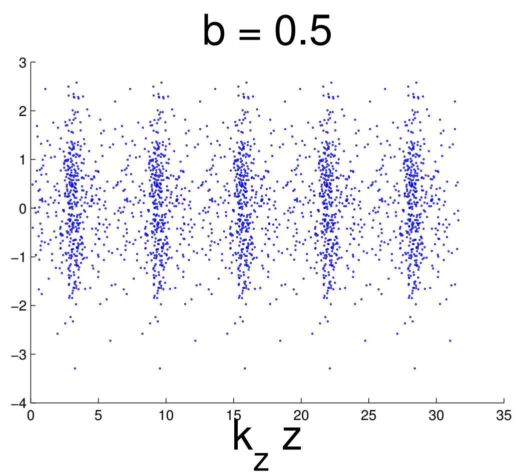
↓

Density Modulation

↓

Coherent Radiation





Working Principle

Resonant Interaction

↓

Energy Modulation

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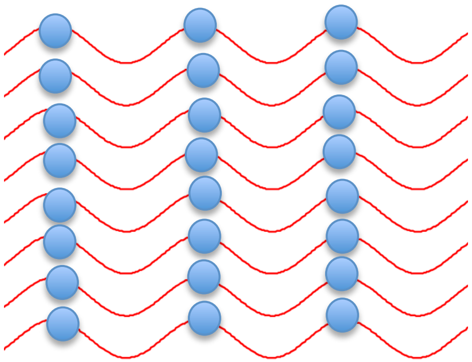
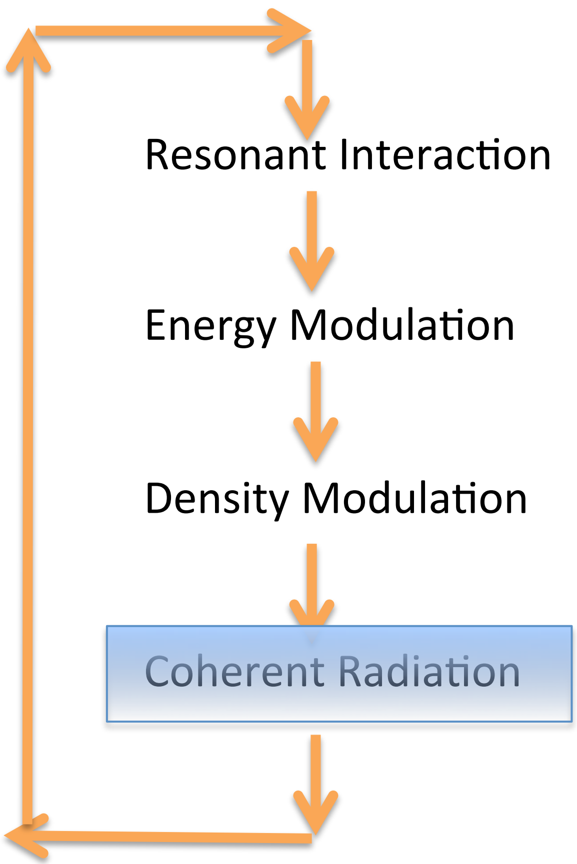
Density Modulation

↓

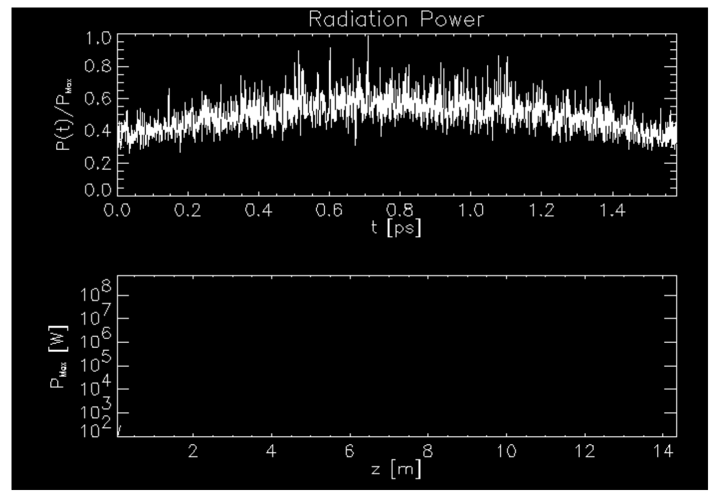
Coherent Radiation



Working Principle



Process goes unstable, leading to exponential growth of power and bunching



Can start from a coherent seed or by noise in electron distribution!

COLLECTIVE INSTABILITIES AND HIGH-GAIN REGIME IN A FREE ELECTRON LASER

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National Synchrotron Light Source, Brookhaven National Laboratory, Upton, NY 11973, USA

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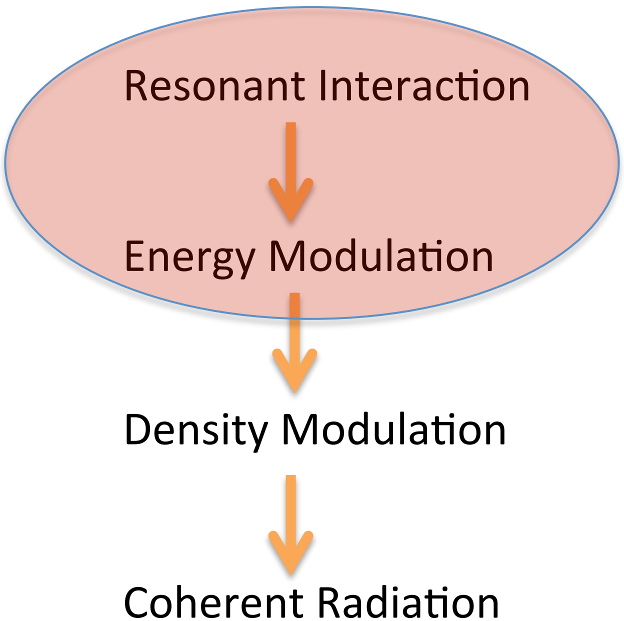
FEL Equations

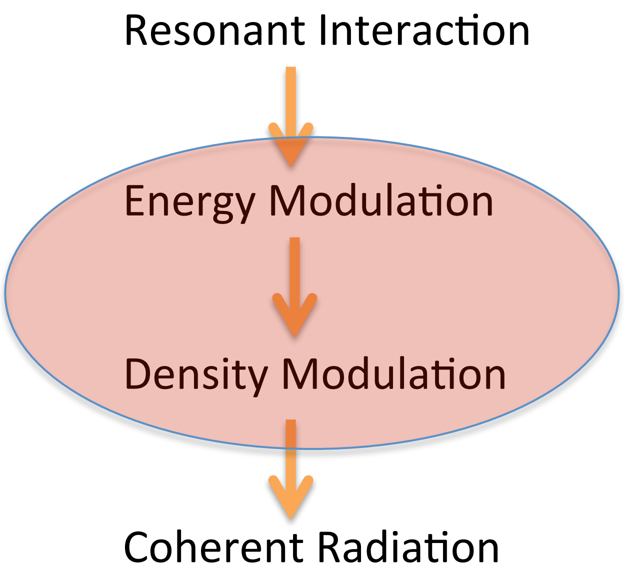


$$\frac{d}{dz}\tilde{\eta} = \frac{eK\bar{E}}{2mc^2\gamma_b^2}$$

$$\eta = \frac{\gamma - \gamma_b}{\gamma_b} \ll 1$$

$$\tilde{\eta} = \frac{1}{N}\sum_{n=1}^{N} \eta_n \exp\left(-i\theta_{n,0}(1+\Delta)\right)$$

FEL Equations



$$\frac{d}{dz}b = -2ik_w\tilde{\eta}$$

FEL Equations

Resonant Interaction

↓

Energy Modulation

↓

Density Modulation

↓

Coherent Radiation

$$(k - k_r)/k$$

$$\left(\frac{d}{dz} + ik_w\Delta\right)\bar{E} = -\frac{eKn_0}{2\gamma_b\epsilon_0}(b)$$

Wave Equation

$$(\nabla^2 - \frac{\partial^2}{c^2 \partial t^2})\vec{A}_p = -\mu_0 \vec{J},$$

$$\vec{E} = -\frac{\partial}{\partial t}\vec{A}_p$$

Good-old inhomogeneous wave equation in Lorenz Gauge

First Approximation: 1-D Limit

$$(\nabla^2 - \frac{\partial^2}{c^2 \partial t^2})\vec{A}_p = -\mu_0 \vec{J},$$

$\vec{E}$ ... omogeneous wave

orenz Gauge

1-Dimensional limit:
Neglect diffraction

$$\nabla^2 \approx \frac{\partial^2}{\partial z^2}$$

2nd Approximation: SVEA

SLAC

SVEA: Slowly Varying Envelope Approximation
Used for narrow bandwidth signals

$$\vec{A}_p = \frac{\hat{x} + i\hat{y}}{2}\tilde{A}_p(z,t)\exp\left(ik_r(z-ct)\right) + c.c.$$

Carrier wave

$$k_r = k_w \frac{2\gamma^2}{1+K^2}$$

2nd Approximation: SVEA

SVEA: Slowly Varying Envelope Approximation
Used for narrow bandwidth signals

$$\vec{A}_p = \frac{\hat{x} + i\hat{y}}{2}\tilde{A}_p(z,t)\exp\left(ik_r(z-ct)\right) + c.c.$$

Slowly varying envelope

$$\frac{\partial}{\partial t}\tilde{A}_p(z,t) \ll ck_r\tilde{A}_p(z,t)$$

$$\frac{\partial}{\partial z}\tilde{A}_p(z,t) \ll k_r\tilde{A}_p(z,t)$$

SVEA

SLAC

$$\frac{\partial^2}{\partial t^2}A_p(z,t) = \left(\frac{\partial^2}{\partial t^2}\tilde{A}_p(z,t) - 2ick_r\frac{\partial}{\partial t}\tilde{A}_p(z,t) - c^2k_r^2\tilde{A}_p(z,t)\right)\exp\left(ik_r(z-ct)\right)$$

$$\frac{\partial^2}{\partial z^2}A_p(z,t) = \left(\frac{\partial^2}{\partial z^2}\tilde{A}_p(z,t) + 2ik_r\frac{\partial}{\partial z}\tilde{A}_p(z,t) - k_r^2\tilde{A}_p(z,t)\right)\exp\left(ik_r(z-ct)\right)$$

SVEA

$$\frac{\partial}{\partial t}\tilde{A}_p(z,t) \ll ck_r\tilde{A}_p(z,t)$$

<<

$$\frac{\partial^2}{\partial t^2}A_p(z,t) = \left(\frac{\partial^2}{\partial t^2}\tilde{A}_p(z,t) - 2ick_r\frac{\partial}{\partial t}\tilde{A}_p(z,t) - c^2k_r^2\tilde{A}_p(z,t)\right)\exp\left(ik_r(z-ct)\right.$$

<<

$$\frac{\partial^2}{\partial z^2}A_p(z,t) = \left(\frac{\partial^2}{\partial z^2}\tilde{A}_p(z,t) + 2ik_r\frac{\partial}{\partial z}\tilde{A}_p(z,t) - k_r^2\tilde{A}_p(z,t)\right)\exp\left(ik_r(z-ct)\right)$$

$$\frac{\partial}{\partial z}\tilde{A}_p(z,t) \ll k_r\tilde{A}_p(z,t)$$

SVEA

$$\left(\nabla^2 - \frac{\partial^2}{c^2 \partial t^2}\right)\vec{A}_p = -\mu_0 \vec{J},$$

$$\frac{\partial^2}{\partial t^2} A_p(z,t) = \Big(\quad - 2ick_r \frac{\partial}{\partial t}\tilde{A}_p(z,t) - c^2 k_r^2 \tilde{A}_p(z,t)\Big) \exp\left(ik_r(z - ct)\right.$$

Cancel out in wave equation

$$\frac{\partial^2}{\partial z^2} A_p(z,t) = \Big(\quad + 2ik_r \frac{\partial}{\partial z}\tilde{A}_p(z,t) - k_r^2 \tilde{A}_p(z,t)\Big) \exp\left(ik_r(z - ct)\right)$$

SVEA

$$(\nabla^2 - \frac{\partial^2}{c^2 \partial t^2}) \vec{A}_p = -\mu_0 \vec{J},$$

$$\frac{\partial^2}{\partial t^2} A_p(z,t) = \Big(\qquad - 2ick_r \frac{\partial}{\partial t} \tilde{A}_p(z,t) \qquad \Big) \exp\left(ik_r(z - ct)\right)$$

$$\frac{\partial^2}{\partial z^2} A_p(z,t) = \Big(\qquad + 2ik_r \frac{\partial}{\partial z} \tilde{A}_p(z,t) \qquad \Big) \exp\left(ik_r(z - ct)\right)$$

Wave Equation

$$2ik_r\left[\left(\frac{\partial}{\partial z}+\frac{\partial}{c\partial t}\right)\tilde{A}_p(z,t)\right]\frac{\hat{x}+i\hat{y}}{2}\exp\left(ik_r(z-ct)\right)+c.c.=-\mu_0\vec{J}.$$

$$\vec{E}=-\frac{\partial}{\partial t}\vec{A}_p\simeq ik_r c\frac{\hat{x}+i\hat{y}}{2}\tilde{A}_p(z,t)\exp\left(ik_r(z-ct)\right)$$

The Current Density

SLAC

$$\vec{j}_\perp = ec\vec{\beta}_\perp \delta(\vec{x} - \vec{x}_n)$$

For each electron

$$\vec{J} = \sum_{n=1}^{N} ec\frac{K}{\gamma_n}\exp(-ik_w z)\frac{\hat{x} + i\hat{y}}{2}\delta_f(z - z_n)\delta_f(\vec{x}_\perp - \vec{x}_{\perp,n}) + c.c.$$

Wave Equation

$$\left(\frac{\partial}{\partial z}+\frac{\partial}{c\partial t}\right)\tilde{E}(z,t)=-\sum_{n=1}^{N}\frac{eK}{2\epsilon_0\gamma_n}\exp(-i\theta)\delta_f(z-z_n)\delta_f(\vec{x}_\perp-\vec{x}_{\perp,n})$$

Wave Equation

$$\left(\frac{\partial}{\partial z} + \frac{\partial}{c\partial t}\right)\tilde{E}(z,t) = -\sum_{n=1}^{N} \frac{eK}{2\epsilon_0\gamma_n} \exp(-i\theta)\delta_f(z - z_n)\delta_f(\vec{x}_\perp - \vec{x}_{\perp,n})$$

$$\theta = (k_r + k_w)z - k_r ct$$

Wave Equation

$$\left(\frac{\partial}{\partial z}+\frac{\partial}{c\partial t}\right)\tilde{E}(z,t)=-\sum_{n=1}^{N}\frac{eK}{2\epsilon_0\gamma_n}\exp(-i\theta)\delta_f(z-z_n)\delta_f(\vec{x}_\perp-\vec{x}_{\perp,n})$$

$$\theta=(k_r+k_w)z-k_rct$$

$$\vec{J}=\sum_{n=1}^{N}ec\frac{K}{\gamma_n}\exp(-ik_wz)\frac{\hat{x}+i\hat{y}}{2}\delta_f(z-z_n)\delta_f(\vec{x}_\perp-\vec{x}_{\perp,n})+c.c.$$

$$2ik_r\left[\left(\frac{\partial}{\partial z}+\frac{\partial}{c\partial t}\right)\tilde{A}_p(z,t)\right]\frac{\hat{x}+i\hat{y}}{2}\exp\left(ik_r(z-ct)\right)+c.c.=-\mu_0\vec{J}.$$

Combination of these two terms

The Ponderomotive Phase

$$\theta = (k_r + k_w)z - k_r ct$$

$$\lambda_r = c\frac{\lambda_w}{v_z} - \lambda_w$$

Combine the two definitions:

$$\theta = k_r(z - v_z t)$$

Theta is a measure of the position along the electron beam

Wave Equation: Change of Variables

$$\frac{\partial}{\partial z} \rightarrow \frac{\partial}{\partial z} + (k_r + k_w)\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{c\partial t} \rightarrow -k_r \frac{\partial}{\partial \theta}$$

$$\left(\frac{\partial}{\partial z} + k_w \frac{\partial}{\partial \theta}\right)\tilde{E}(z,\theta) = -\sum_{n=1}^{N} \frac{eK}{2\epsilon_0 \gamma_n} \exp(-i\theta)\delta_f(z - z_n)\delta_f(\vec{x}_\perp - \vec{x}_{\perp,}$$

Wave Equation: Frequency Domain

$$\left(\frac{\partial}{\partial z} + k_w \frac{\partial}{\partial \theta}\right) \tilde{E}(z,\theta) = -\sum_{n=1}^{N} \frac{eK}{2\epsilon_0 \gamma_n} \exp(-i\theta)\delta_f(z - z_n)\delta_f(\vec{x}_\perp - \vec{x}_\perp$$

$$\tilde{E}(z,\theta) = \bar{E}(z,\Delta)\exp(i\Delta\theta)$$

$$k_w \frac{\partial}{\partial \theta} \rightarrow k_w i\Delta$$

$$\Delta = \frac{k - k_r}{k_r}$$

Take a volume average

↓

$$\left(\frac{\partial}{\partial z} + k_w i\Delta\right)\bar{E} = -\frac{eKn_0}{2\epsilon_0}\frac{1}{N}\sum_{n=1}^{N}\frac{\exp\left(-i\theta_n(1+\Delta)\right)}{\gamma_n}$$

Wave Equation: Frequency Domain

$$\left(\frac{\partial}{\partial z} + k_w \frac{\partial}{\partial \theta}\right) \tilde{E}(z,\theta) = -\sum_{n=1}^{N} \frac{eK}{2\epsilon_0 \gamma_n} \exp(-i\theta)\delta_f(z - z_n)\delta_f(\vec{x}_\perp - \vec{x}_\perp$$

$$\tilde{E}(z,\theta) = \bar{E}(z,\Delta)\exp(i\Delta\theta)$$

$$k_w \frac{\partial}{\partial \theta} \rightarrow k_w i \Delta$$

$$\Delta = \frac{k - k_r}{k_r}$$

Take a volume average

Beam volume density
#particles/volume

$$\left(\frac{\partial}{\partial z} + k_w i \Delta\right) \bar{E} = -\frac{eKn_0}{2\epsilon_0}\frac{1}{N}\sum_{n=1}^{N} \frac{\exp\left(-i\theta_n(1+\Delta)\right)}{\gamma_n}$$

Wave Equation and the Bunching Factor

$$\left(\frac{\partial}{\partial z} + k_w i\Delta\right)\bar{E} = -\frac{eKn_0}{2\epsilon_0}\frac{1}{N}\sum_{n=1}^{N}\frac{\exp\left(-i\theta_n(1+\Delta)\right)}{\gamma_n}$$

Quasi monoenergetic beam $\gamma_n \approx \gamma_b$

$$\left(\frac{d}{dz} + ik_w\Delta\right)\bar{E} = -\frac{eKn_0}{2\gamma_b\epsilon_0}\, b$$

$$b = \frac{1}{N}\sum_{n=1}^{N}\exp\left(-i\theta_n(1+\Delta)\right)$$

How Does b evolve?

$$\frac{d}{dz}b = -i(1+\Delta)\frac{1}{N}\sum_{n=1}^{N}\frac{d}{dz}\theta_n \exp\left(-i\theta_n \quad (1+\Delta)\right)$$

$$b = \frac{1}{N}\sum_{n=1}^{N}\exp\left(-i\theta_n(1+\Delta)\right)$$

How Does b evolve?

$$\frac{d}{dz}b = -i(1+\Delta)\frac{1}{N}\sum_{n=1}^{N}\frac{d}{dz}\theta_n \exp\left(-i\theta_n \quad (1+\Delta)\right)$$

Need an equation for this!

How Does b evolve?

$$\frac{d\theta}{dz} = k_r + k_w - \frac{k_r}{\beta_z} = k_r + k_w - \frac{k_r}{1 - \frac{1+K^2}{\gamma^2}}$$

Just substitute β_z

Use definition of resonant frequency

$$= k_w \left(1 - \frac{\gamma_b^2}{\gamma^2}\right) \approx 2k_w \eta$$

Quasi monochromatic beam

$$\eta = \frac{\gamma - \gamma_b}{\gamma_b} << 1$$

Bunching Equation

$$\frac{db}{dz} \approx 2ik_w \frac{1}{N} \sum \eta_n e^{-i(1+\Delta)\theta_n}$$

Energy Modulation!

Bunching Equation

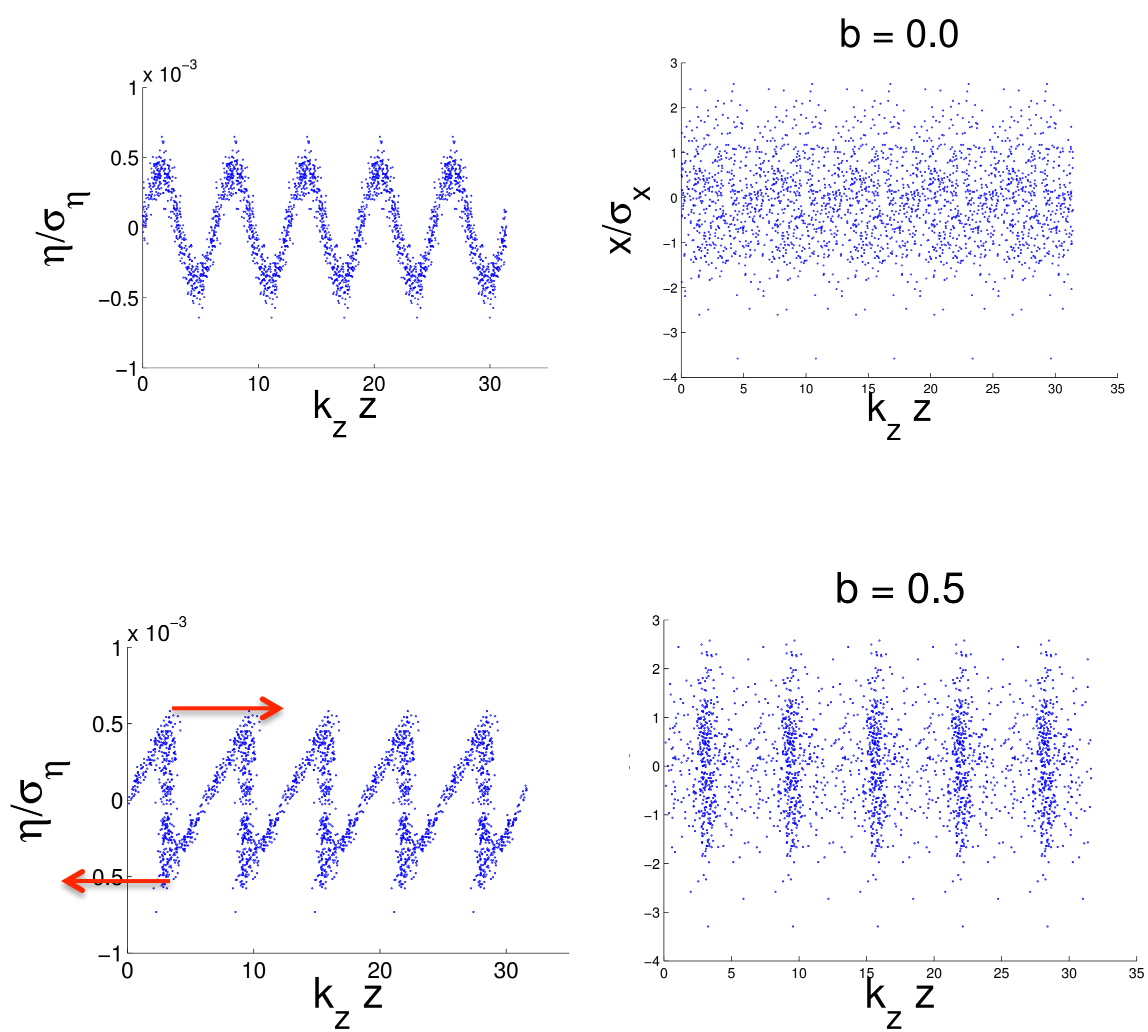
$$\frac{db}{dz} \approx 2ik_w \frac{1}{N} \sum \eta_n e^{-i(1+\Delta)\theta_n}$$

Energy Modulation!

$$\frac{db}{dz} \approx -2ik_w \tilde{\eta}$$

From Bunching to Energy Modulation

$$\frac{db}{dz} \approx -2ik_w\tilde{\eta}$$



Energy Modulation Equation

$$mc^2 \frac{d}{dz}\gamma_n = e\vec{\beta} \cdot \vec{E}(z,t)$$

$$\frac{d}{dz}\eta_n = \frac{eK\bar{E}}{2mc^2\gamma_b\gamma_n} \exp\left(i\theta_n(1+\Delta)\right) + c.c.$$

Energy Modulation Equation

$$\frac{d}{dz}\eta_n = \frac{eK\bar{E}}{2mc^2\gamma_b\gamma_n}\exp\left(i\theta_n(1+\Delta)\right) + c.c.$$

$$\frac{d\tilde{\eta}}{dz} = \frac{d}{dz}\frac{1}{N}\sum \eta_n e^{-i(1+\Delta)\theta_n} = \frac{1}{N}\sum\left(\frac{d\eta_n}{dz}\right)e^{-i(1+\Delta)\theta_n} + -\frac{i(1+\Delta)}{N}\sum \eta_n \frac{d\theta_n}{dz}e^{-i(1+\Delta)\theta_n}$$

Energy Modulation Equation

$$\frac{d}{dz}\eta_n = \frac{eK\bar{E}}{2mc^2\gamma_b\gamma_n}\exp\left(i\theta_n(1+\Delta)\right) + c.c.$$

$$\frac{d\tilde{\eta}}{dz} = \frac{d}{dz}\frac{1}{N}\sum \eta_n e^{-i(1+\Delta)\theta_n} = \frac{1}{N}\sum\left(\frac{d\eta_n}{dz}\right)e^{-i(1+\Delta)\theta_n} + -\frac{i(1+\Delta)}{N}\sum \eta_n \frac{d\theta_n}{dz} e^{-i(1+\Delta)\theta_n}$$

CAREFUL: 2ND ORDER TERM!

NEEDS A WHOLE NEW HIERARCHY OF EQUATIONS TO CLOSE THE SYSTEM!

Energy Modulation Equation

$$\frac{d}{dz}\eta_n = \frac{eK\bar{E}}{2mc^2\gamma_b\gamma_n}\exp\left(i\theta_n(1+\Delta)\right) + c.c.$$

$$\frac{d\tilde{\eta}}{dz} = \frac{d}{dz}\frac{1}{N}\sum \eta_n e^{-i(1+\Delta)\theta_n} = \frac{1}{N}\sum\left(\frac{d\eta_n}{dz}\right)e^{-i(1+\Delta)\theta_n} + -\frac{i(1+\Delta)}{N}\sum \eta_n \frac{d\theta_n}{dz}e^{-i(1+\Delta)\theta_n}$$

HERE WE MAKE THE LINEAR APPROXIMATION!

b, E and $\tilde{\eta}$

are small perturbations. Neglect all higher order terms

Energy Modulation Equation

SLAC

$$\frac{d}{dz}\eta_n = \frac{eK\bar{E}}{2mc^2\gamma_b\gamma_n}\exp\left(i\theta_n(1+\Delta)\right) + c.c.$$

$$\frac{d\tilde{\eta}}{dz} = \frac{d}{dz}\frac{1}{N}\sum \eta_n e^{-i(1+\Delta)\theta_n} = \frac{1}{N}\sum\left(\frac{d\eta_n}{dz}\right)e^{-i(1+\Delta)\theta_n}$$

Linear approximation allows us
to close the system!!

$$\frac{d}{dz}\tilde{\eta} \simeq \frac{eK\bar{E}}{2mc^2\gamma_b^2}$$

Linear FEL Equations!

$$\left(\frac{d}{dz} + ik_w\Delta\right)\bar{E} = -\frac{eKn_0}{2\gamma_b\epsilon_0}(b)$$

$$\frac{d}{dz}b = -2ik_w\tilde{\eta},$$

$$\frac{d}{dz}\tilde{\eta} = \frac{eK\bar{E}}{2mc^2\gamma_b^2}.$$

15 MINUTE BREAK!

Assumptions

- Neglect diffraction
- Small signal (b << 1)
- Slowly varying envelope (i.e. narrow bandwidth signal)
- No velocity spread (longitudinal and transverse)

Equilibrium Condition

If particles are uniformly distributed: b = 0
Initial field = 0
Beam perfectly monoenergetic: $\tilde{\eta} = 0$

$$\left(\frac{d}{dz} + ik_w\Delta\right)\bar{E} = 0$$

$$\frac{d}{dz}b = 0$$

$$\frac{d}{dz}\tilde{\eta} = 0$$

System at equilibrium

FEL Instability

Is the equilibrium stable
or unstable?

$$\Delta = 0$$

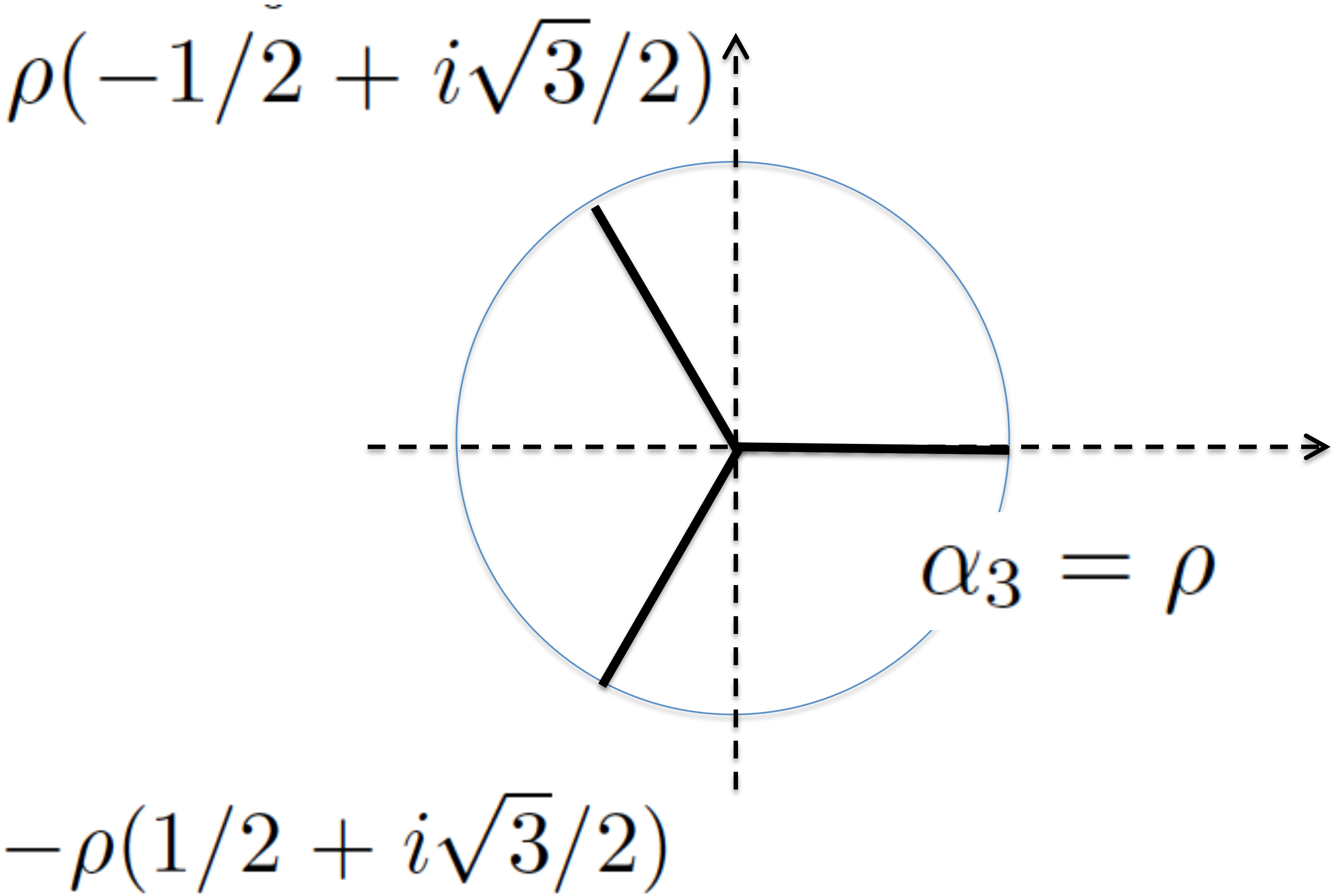
$$\bar{E}, b, \tilde{\eta} \propto \exp(-2ik_w \alpha z)$$

↓

$$\alpha^3 = \rho^3$$

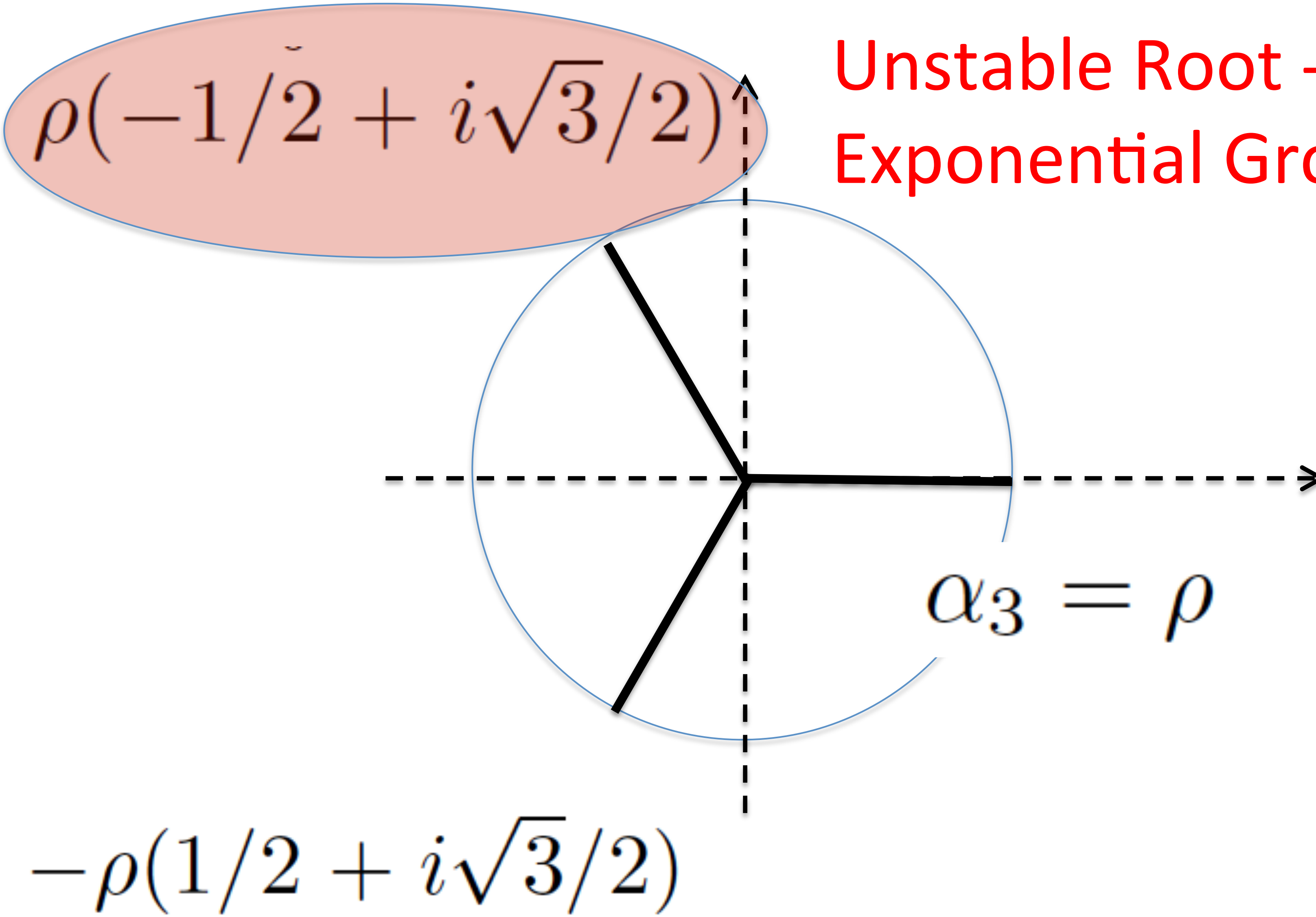
$$\rho = (K k_p / 4k_w)^{(2/3)} \qquad k_p^2 = n_0 e^2 / \epsilon_0 \gamma_b^3 m c^2$$

Roots



Roots

Unstable Root -> Exponential Growth



The ρ parameter

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$$\rho = (Kk_p/4k_w)^{(2/3)}$$

$\propto n_e^{1/3}$

High density -> higher gain!
(note: scaling typical of all 3-wave instabilities…)

$\propto 1/\gamma$

Smaller growth rate at higher energies

$\propto K^{2/3}$

Stronger magnetic field -> higher gain

Typically 10^{-3} to 10^{-4} for x-ray parameters

The gain-length

What theorists call gain-length:

$$l_g = \frac{1}{2k_w\rho}$$

Because it makes equations look pretty…

What experimentalists call gain-length:

$$L_g = \frac{1}{2\sqrt{3}k_w\rho}$$

Because power is what you measure…

$$P \propto \exp\left(\frac{z}{L_g}\right)$$

That’s Pretty Much it…

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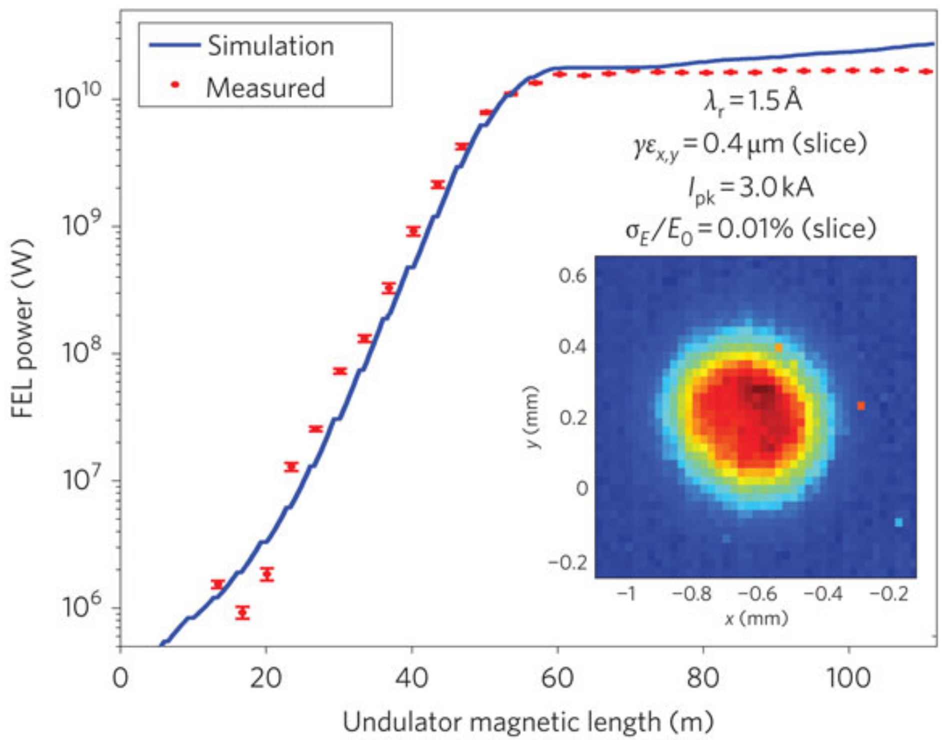


nature photonics

ARTICLES

PUBLISHED ONLINE: 1 AUGUST 2010 | DOI: 10.1038/NPHOTON.2010.176

First lasing and operation of an ångstrom-wavelength free-electron laser

P. Emma[1]*, R. Akre[1], J. Arthur[1], R. Bionta[2], C. Bostedt[1], J. Bozek[1], A. Brachmann[1], P. Bucksbaum[1], R. Coffee[1], F.-J. Decker[1], Y. Ding[1], D. Dowell[1], S. Edstrom[1], A. Fisher[1], J. Frisch[1], S. Gilevich[1], J. Hastings[1], G. Hays[1], Ph. Hering[1], Z. Huang[1], R. Iverson[1], H. Loos[1], M. Messerschmidt[1], A. Miahnahri[1], S. Moeller[1], H.-D. Nuhn[1], G. Pile[3], D. Ratner[1], J. Rzepiela[1], D. Schultz[1], T. Smith[1], P. Stefan[1], H. Tompkins[1], J. Turner[1], J. Welch[1], W. White[1], J. Wu[1], G. Yocky[1] and J. Galayda[1]

What Happens at Saturation?

SLAC

$$\frac{d}{dz}b = -2ik_w\tilde{\eta}$$

$$|\tilde{\eta}| = \rho|b|$$

@ saturation b ~ 1 → $|\tilde{\eta}_{sat}| \sim \rho$

What Happens at Saturation?

$$P_{rad} = Z_0|\bar{E}|^2 = \rho P_b |b|^2$$

@ saturation b ~ 1

$$P_{sat} \sim \rho P_b$$

For LCLS that's ~10-100 GW

$$P_b = n_0 \gamma m c^3$$

Electron beam power density

Wait a Minute…

$$P_{sat} \approx \rho P_b \propto I^{4/3}$$

But I promised you that coherent radiation goes like square of # of particles…

Wait a Minute…

SLAC

$$P_{sat} \approx \rho P_b \propto I^{4/3}$$

But I promised you that coherent radiation goes like square of # of particles…

What matters is # of particles in a slippage length!

$$N^2_{SLIPPAGE} = \left(\frac{I}{e} \frac{\lambda_r L_{sat}}{\lambda_w} \right)^2$$

$$L_{sat} \propto \frac{1}{\rho} \propto I^{-1/3}$$

$$N^2_{SLIPPAGE} \propto I^{4/3}$$

Normalized FEL Equations

SLAC

Normalize everything to saturation value

$$\frac{d}{d\bar{z}}a + i\frac{\Delta}{2\rho}a = -b$$

$$\frac{d}{d\bar{z}}b = -ip$$

$$\frac{d}{d\bar{z}}p = a$$

$$p = \frac{\tilde{\eta}}{\rho}$$

$$a = \bar{E}\sqrt{\frac{Z_0}{P_b}}$$

$$\bar{z} = 2k_w\rho z$$

Normalized FEL Equations

SLAC

Natural scaling of detuning is also ~ ρ

Normalize everything to saturation value

$$\frac{d}{d\bar{z}}a + i\frac{\Delta}{2\rho}a = -b$$

$$\frac{d}{d\bar{z}}b = -ip$$

$$\frac{d}{d\bar{z}}p = a$$

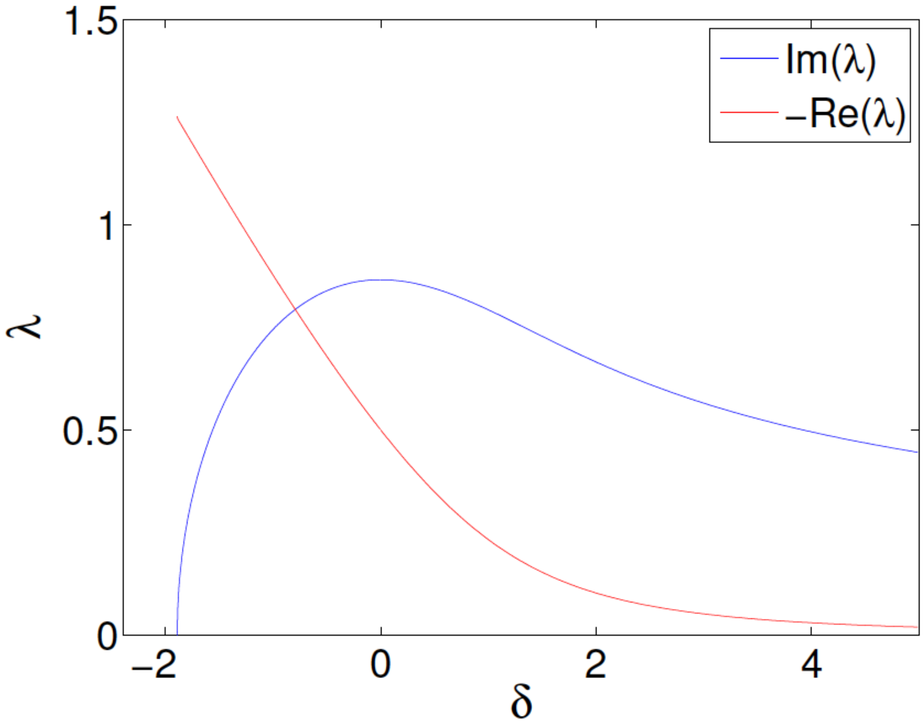
$$p = \frac{\tilde{\eta}}{\rho}$$

$$a = \bar{E}\sqrt{\frac{Z_0}{P_b}}$$

$\delta = \Delta/2\rho$

Dispersion Relation for General Detuning

$$\lambda^3 - \delta\lambda^2 - 1 = 0$$



Assume all quantities

$$\propto \exp\left(i\lambda\bar{z}\right)$$

Substitute into FEL linear equations

To 2nd Order…

$$\lambda_1 = -\frac{1}{2} + \frac{\delta}{3} - \frac{\delta^2}{18} + i\frac{\sqrt{3}}{2}\left(1 - \frac{\delta^2}{9}\right)$$

$$\frac{d}{d\delta}\lambda = \frac{\lambda}{3\lambda - 2\delta}$$

$$\frac{d^2}{d\delta^2}\lambda = \frac{\frac{d}{d\delta}\lambda}{3\lambda - 2\delta} - \frac{\lambda(3\frac{d}{d\delta}\lambda - 2)}{(3\lambda - 2\delta)^2}$$

To 2nd Order…

SLAC

$$\lambda_1 = -\frac{1}{2} + \frac{\delta}{3} - \frac{\delta^2}{18} + i\frac{\sqrt{3}}{2}\left(1 - \frac{\delta^2}{9}\right)$$

$$\sigma_\omega/\omega = 6\rho/\sqrt{(2\sqrt{3}\bar{z})}$$

Bandwidth ~ z ½

@ saturation (z ~ 10 to 20)

$\Delta\omega/\omega \sim \rho$

To 2nd Order…

$$\lambda_1 = -\frac{1}{2} + \frac{\delta}{3} - \frac{\delta^2}{18} + i\frac{\sqrt{3}}{2}\left(1 - \frac{\delta^2}{9}\right)$$

Group Velocity = v_b + 1/3 slippage rate

Initial Value Problem

$$a = \sum_{j=1}^{3} \frac{-i}{\frac{d}{d\lambda} D|_{\lambda=\lambda_j}} \exp(-i\lambda\bar{z})(i\lambda_j^2 a_0 + \lambda_j b_0 + p_0)$$

Initial values of three variables

$$D(\lambda, \delta) = \lambda^3 - \delta\lambda^2 - 1$$

FEL can be triggered by either

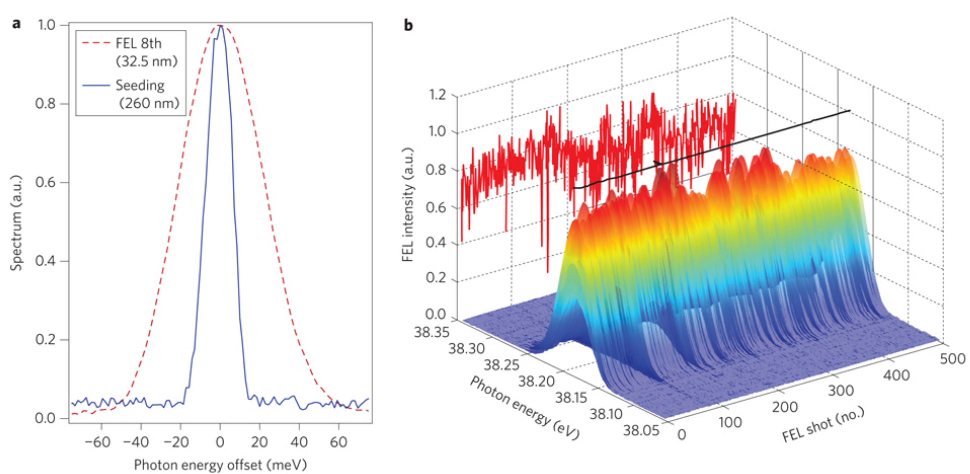
- an initial radiation field
- an initial microbunching
- an initial energy modulation

Experimentally, at x-rays it's difficult to generate a starting value for any of these quantities

Seeded Free-Electron Laser

$$a = \sum_{j=1}^{3} \frac{-i}{\frac{d}{d\lambda} D|_{\lambda=\lambda_j}} \exp(-i\lambda\bar{z})(i\lambda_j^2 a_0 + \lambda_j b_0 + c_0)$$

An initial narrow bandwidth laser is used to initiate the process
-> Narrow-bandwidth coherent pulse amplified to saturation



Shot-Noise

Seeding can't be done at x-rays: NO LASERS!

Luckily nature gives us a natural initial value for beam microbunching: NOISE

$$< b_{sn} > = 0$$

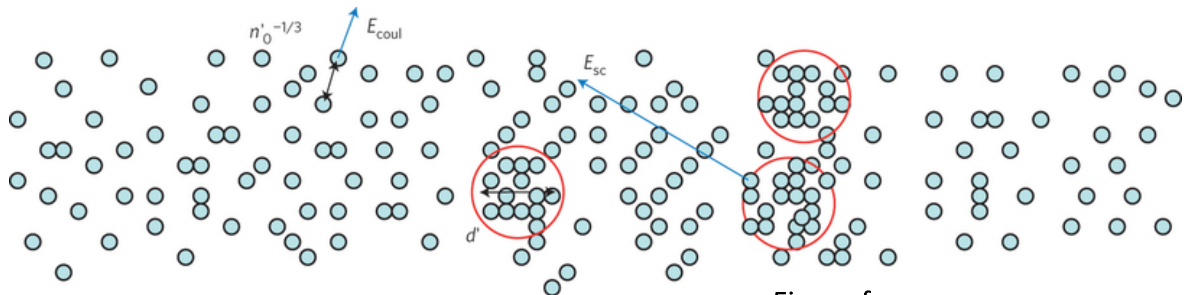


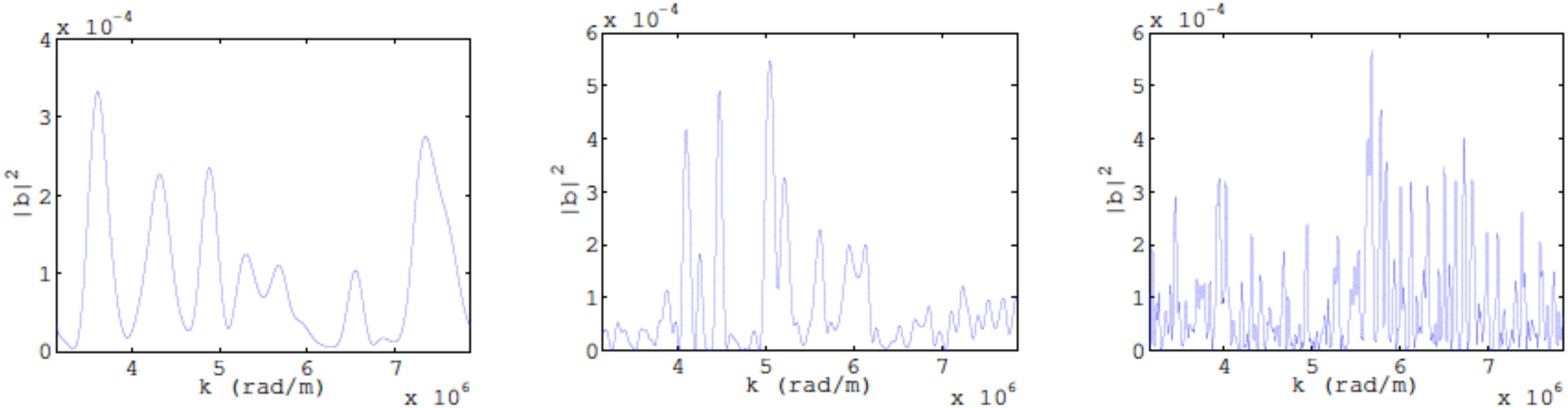
Figure from:
Avraham Gover et al.
Nature Physics **8**, 877–880 (2012)

$$< \left| b_{sn} \right|^2 > = \frac{1}{N^2} \sum \exp(-i\theta_n) \sum \exp(i\theta_m)$$

$$= \frac{1}{N^2} \sum_{n=m} \exp\left(-i(\theta_n - \theta_m)\right) + \frac{1}{N^2} \sum_{n \neq m} \exp\left(-i(\theta_n - \theta_m)\right)$$

$$= \frac{1}{N}$$

Shot-Noise Microbunching In Frequency Domain



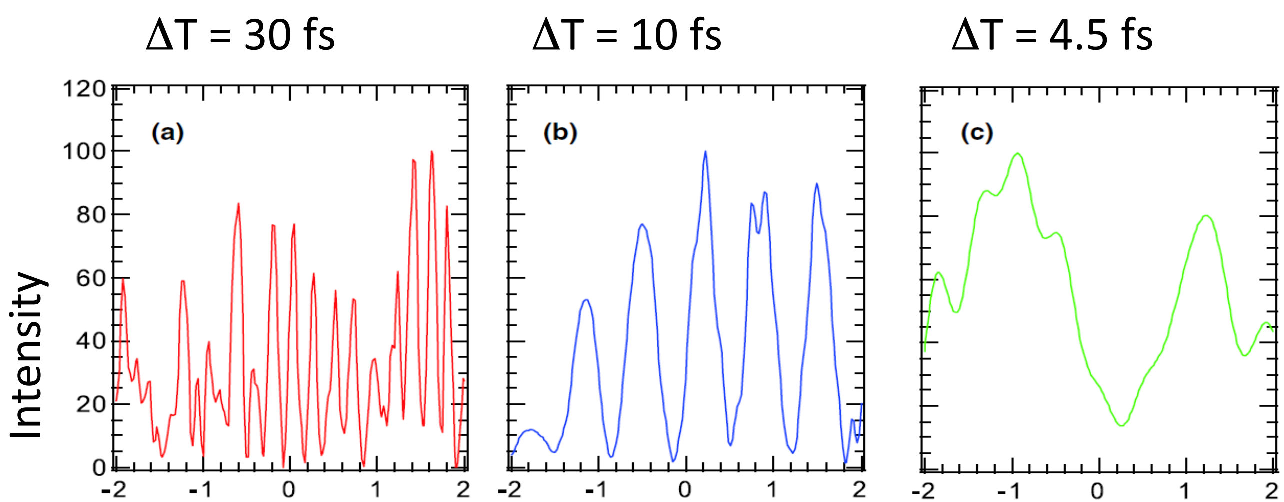
Increasing bunch length:
Narrower spikes

Shot-Noise Microbunching

$$< b(k)b^*(k') >= \frac{1}{N}F(k-k')$$

Spectral autocorrelation ~ Fourier transform of longitudinal distribution at k-k'

(Nice derivation in Saldin's book!)



ΔPhoton Energy

PRL 109, 144801 (2012) PHYSICAL REVIEW LETTERS week ending 5 OCTOBER 2012

Determination of the Pulse Duration of an X-Ray Free Electron Laser Using Highly Resolved Single-Shot Spectra

Yuichi Inubushi,[1,*] Kensuke Tono,[2] Tadashi Togashi,[2] Takahiro Sato,[1] Takaki Hatsui,[1] Takashi Kameshima,[2] Kazuaki Togawa,[1] Toru Hara,[1] Takashi Tanaka,[1] Hitoshi Tanaka,[1] Tetsuya Ishikawa,[1] and Makina Yabashi[1]

Self Amplified Spontaneous Emission

$$a(\bar{z},\delta) = G(\bar{z},\delta) b_0$$

From initial value problem

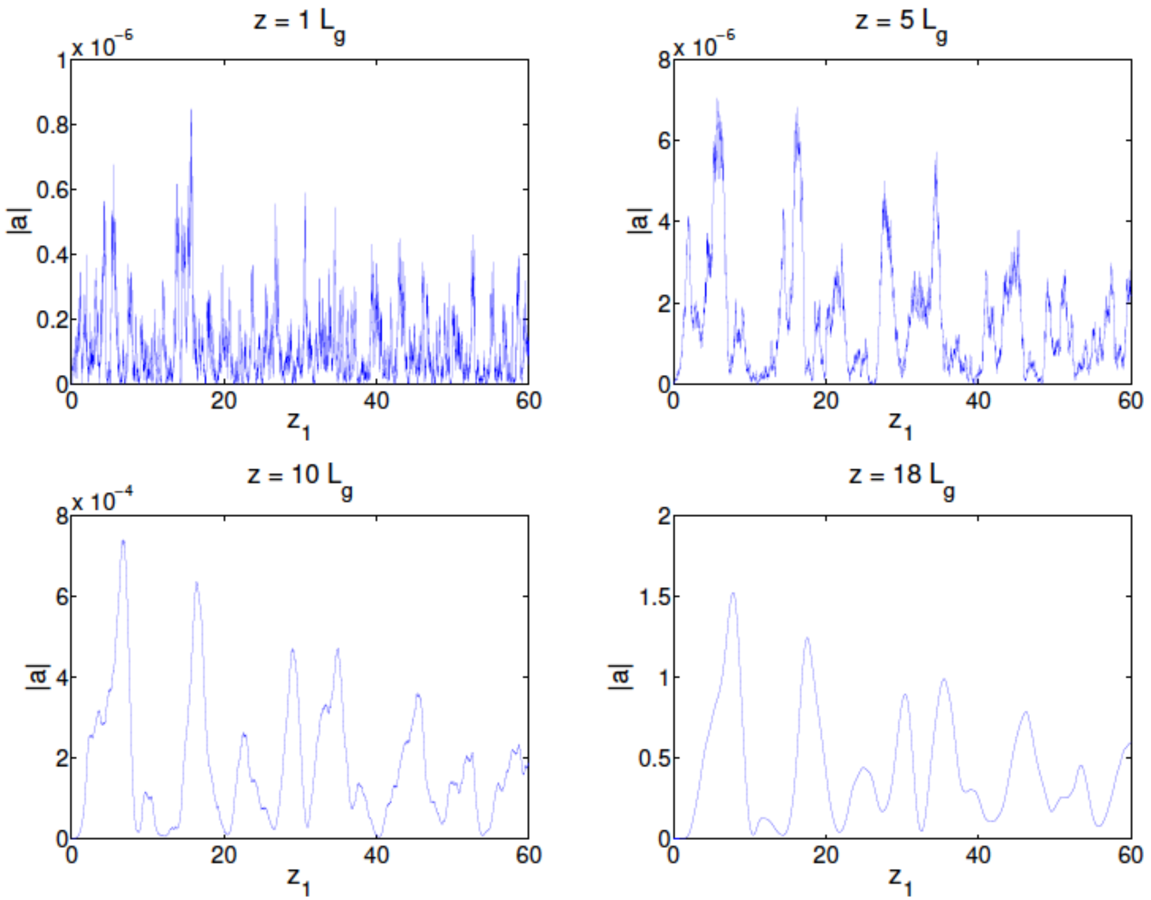
$$G(\bar{z},\delta) = \frac{-i}{3\lambda - 2\delta} \exp(-i\lambda\bar{z})$$

In SASE b_0 is shot-noise microbunching

$$< b_{sn} > = 0$$

$$< |b_{sn}|^2 > = \frac{1}{N}$$

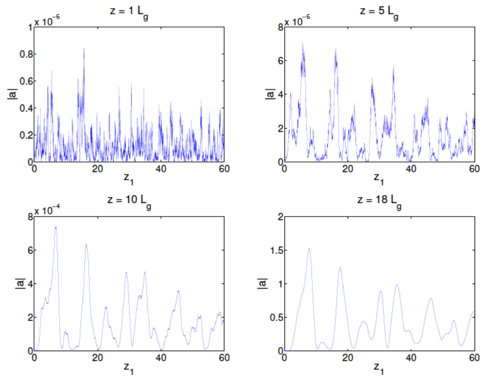
What Does SASE Look Like?



Spiky temporal structure.

Spikes get broader as radiation slips across the electron bunch!

The Cooperation Length



$$\bar{a}(\bar{z}, \bar{z}_1) = \frac{k_r L_b \rho}{\pi} \int \exp(i\delta\bar{z}_1) b_0(\delta) G(\bar{z}, \delta) d\delta$$

Note: rho defines the spectral width

$$\delta = \Delta / 2\rho$$

The Fourier conjuate variable is

$$\bar{z}_1 = 2\rho\theta = 2k_r\rho(z - v_z t)$$

Which means that the length-scale of the SASE spikes is

$$l_c = 1/2k_r\rho$$

"Cooperation length" =
slippage in a gain-length

What is the Average Power?

SLAC

$$\bar{a}(\bar{z},\bar{z}_1) = \frac{k_r L_b \rho}{\pi} \int \exp(i\delta\bar{z}_1) b_0(\delta) G(\bar{z},\delta) d\delta$$

We can use Parseval's theorem to compute average power

$$< |\bar{a}(\bar{z},\bar{z}_1)|^2 >= \frac{k_r L_b \rho}{N\pi} \int |G(\bar{z},\delta)|^2 d\delta$$

Equivalent Shot-Noise Power

SLAC

Approximate solution by neglecting δ dependence of residue term:
Gain function turns into a Gaussian!

$$P_{SASE} = \frac{1}{9} P_{sn} \exp\left(2\rho k_w \sqrt{3} z\right)$$

$$P_{sn} = P_b \frac{6\rho^2}{N_\lambda} \sqrt{\frac{\pi}{\bar{z}\sqrt{3}}}$$

N_λ number of particles in a wavelength

~few to tens of kW for typical x-ray FELs

Using Our 1-D Theory…

$$< \bar{a}(\bar{z}_1)\bar{a}^*(\bar{z}_1 + \bar{z}_1') >= \left(\frac{k_r L_b \rho}{N\pi}\right) \int \exp(-i\delta \bar{z}_1') |G(\bar{z}, \delta')|^2 d\delta$$

Wiener's theorem:
Autocorrelation function = Fourier Transform of spectral power density

Using the same Gaussian approximation:

$$< \bar{a}(\bar{z}, \bar{z}_1)\bar{a}^*(\bar{z}, \bar{z}_1 + \bar{z}_1') >=< |\bar{a}(\bar{z}, \bar{z}_1)|^2 > \exp\left(-\frac{\bar{z}_1'^2}{2\sigma_{\bar{z}_1,c}^2}\right)$$

$$\sigma_{\bar{z}_1,c} = \frac{\sqrt{2\sqrt{3}\bar{z}}}{3}$$

Coherence length grows as a function of time!
(Consistently with our intuition from previous slide…)

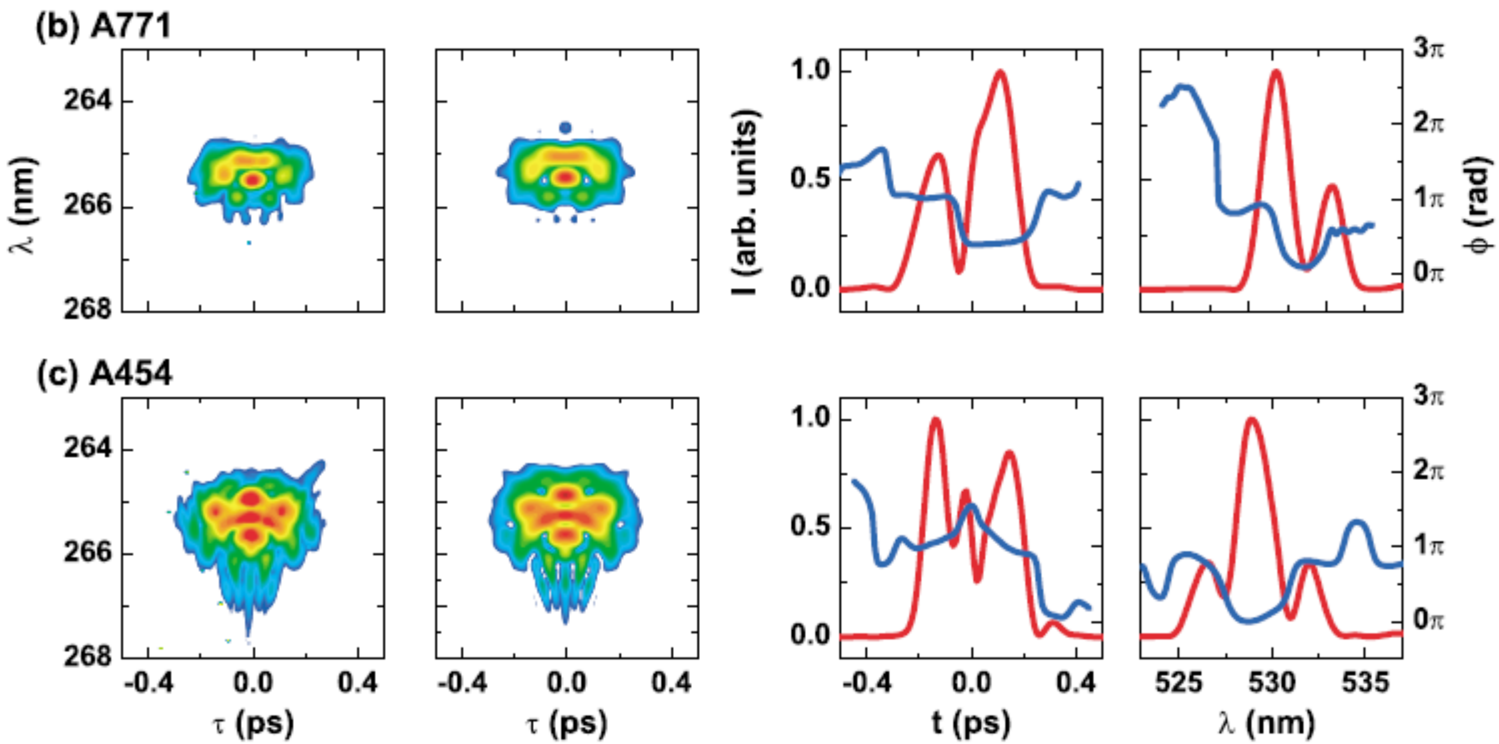
Coherence length

$$\sigma_{\bar{z}_1,c} = \frac{\sqrt{2\sqrt{3}\bar{z}}}{3}$$

Which means at saturation (10-20 gain-lengths)

RMS coherence length ~ 1 cooperation length

SASE Spikes: Experimental Observation

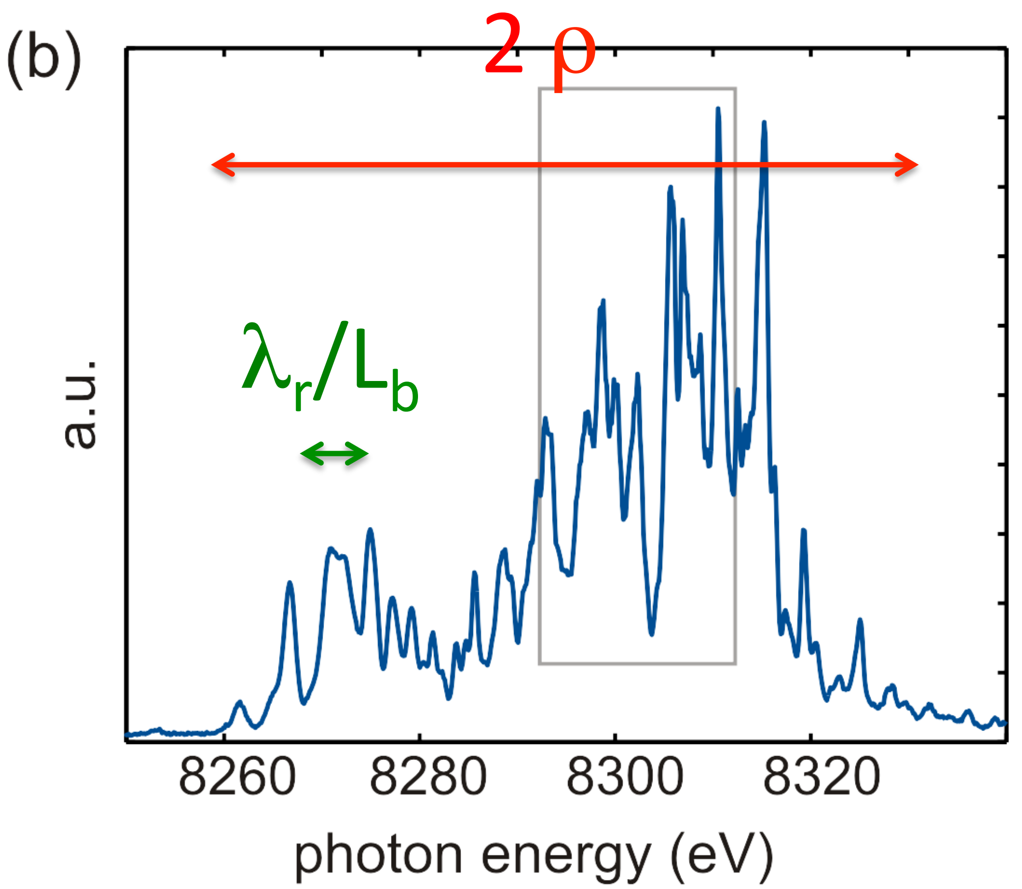


VOLUME 91, NUMBER 24 PHYSICAL REVIEW LETTERS week ending 12 DECEMBER 2003

Characterization of a Chaotic Optical Field Using a High-Gain, Self-Amplified Free-Electron Laser

Yuelin Li,[1,*] Samuel Krinsky,[2] John W. Lewellen,[1] Kwang-Je Kim,[1] Vadim Sajaev,[1] and Stephen V. Milton[1]

SASE Spikes: Spectral Measurements



Bibliography

COLLECTIVE INSTABILITIES AND HIGH-GAIN REGIME IN A FREE ELECTRON LASER

R. BONIFACIO *, C. PELLEGRINI
National Synchrotron Light Source, Brookhaven National Laboratory, Upton, NY 11973, USA

and

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Physics Department, Drexel University, Philadelphia, PA 19104, USA

Received 5 April 1984

VOLUME 73, NUMBER 1 PHYSICAL REVIEW LETTERS 4 JULY 1994

Spectrum, Temporal Structure, and Fluctuations in a High-Gain Free-Electron Laser Starting from Noise

R. Bonifacio,[1,2] L. De Salvo,[1] P. Pierini,[2] N. Piovella,[1] and C. Pellegrini[3]

Non-Ideal Effects

SLAC

Energy-spread:

Electron beams are not mono-energetic but have a small spread.
How much spread can we tolerate?

$$\frac{d\theta}{dz} = 2k_w\eta$$

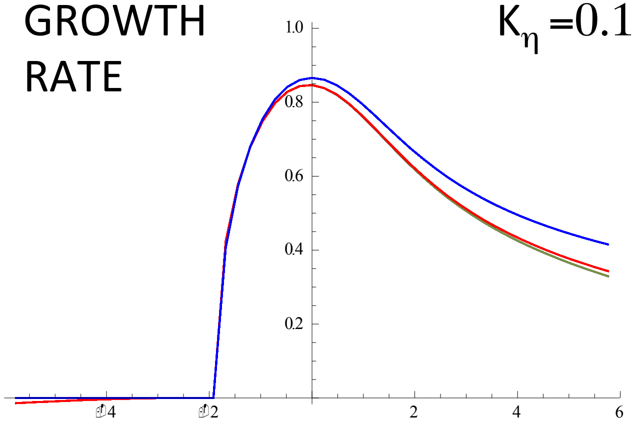
$$\sigma_\theta = 2k_w\sigma_\eta \mathrm{l_g} = \frac{\sigma_\eta}{\rho} << 1$$

Rho is the energy acceptance of the FEL!

Dispersion Relation

GROWTH RATE

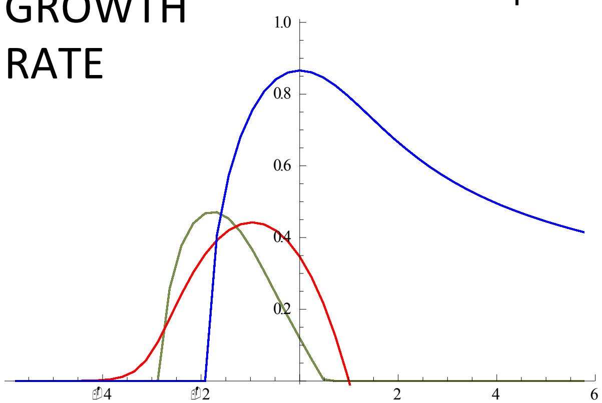
$K_\eta = 0.15$



Detuning

$K_\eta = 1$

GROWTH RATE



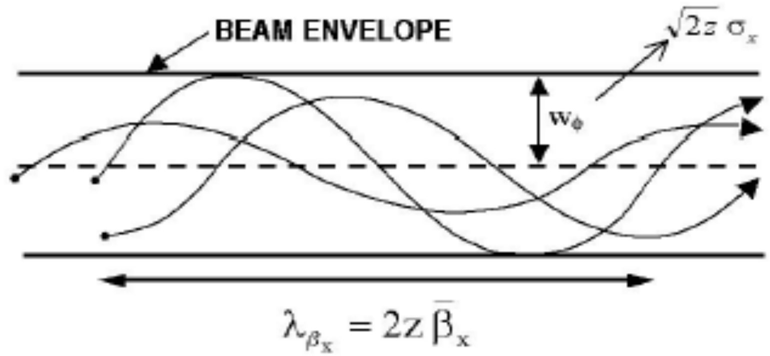
Detuning

Cold
Gaussian
Laser Heater

Whatever your distribution, you want
Energy-Spread << ρ

Emittance

Electrons have finite transverse velocity spre



$$\sigma^2_{\vec{\beta}_\perp} = \frac{\varepsilon}{\beta_f}$$

$$v_z = c\sqrt{1 - \frac{1}{\gamma^2} - \vec{\beta}_\perp^2} \approx 1 - \frac{1}{2\gamma^2} - \frac{\vec{\beta}_\perp^2}{2}$$

Spread in transverse velocity =
Spread in longitudinal velocity!

Phase-spread in a gain-length <<1

$$k_r \frac{\varepsilon}{\beta_f} l_g << 1$$

Long-story short: you need small emittance for FEL!!

Diffraction Losses

The lengt-scale for radiation to diffract out of the beam is the Rayleigh length:

$$L_r = \frac{1}{2} k_r \sigma_x^2$$

Diffraction losses are negligible if

$$L_r >> l_g$$

If You Thought This Was Complicated...

The full-blown 3-D theory can account for all these effects simultaneously...

$$\left(\mu - \bar{\nu} + \frac{\bar{\boldsymbol{\nabla}}_\perp^2}{2}\right) A(\bar{\boldsymbol{x}}) - \frac{1}{2\pi\bar{\sigma}_x^2}\int_{-\infty}^{0} \tau d\tau \frac{e^{-\bar{\sigma}_\eta^2\tau^2/2 - i\mu\tau}}{\sin^2(\bar{k}_\beta\tau)} \int d^2\bar{x}' A(\bar{\boldsymbol{x}}') \\ \times \exp\left[-\frac{\bar{\boldsymbol{x}}^2 + (\bar{\boldsymbol{x}}')^2 - 2\bar{\boldsymbol{x}}\cdot\bar{\boldsymbol{x}}'\cos(\bar{k}_\beta\tau)}{2\sin^2(\bar{k}_\beta\tau)}\left(i\bar{k}_\beta^2\tau + \frac{1}{\bar{\sigma}_x^2}\right)\right] = 0.$$

No Panic!

IT LOOKS UGLY BUT…

The dispersion relation can be expressed as a function of 4 dimensionless parameters

$$\left(\mu - \bar{\nu} + \frac{\bar{\boldsymbol{\nabla}}_\perp^2}{2}\right) A(\bar{\boldsymbol{x}}) - \frac{1}{2\pi\bar{\sigma}_x^2}\int_{-\infty}^{0} \tau d\tau \frac{e^{-\bar{\sigma}_\eta^2\tau^2/2 - i\mu\tau}}{\sin^2(\bar{k}_\beta\tau)} \int d^2\bar{x}' A(\bar{\boldsymbol{x}}')$$

$$\times \exp\left[-\frac{\bar{\boldsymbol{x}}^2 + (\bar{\boldsymbol{x}}')^2 - 2\bar{\boldsymbol{x}}\cdot\bar{\boldsymbol{x}}'\cos(\bar{k}_\beta\tau)}{2\sin^2(\bar{k}_\beta\tau)}\left(i\bar{k}_\beta^2\tau + \frac{1}{\bar{\sigma}_x^2}\right)\right] = 0.$$

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The dispersion relation can be expressed as a function of 4 dimensionless parameters

$$\left(\mu - \bar{\nu} + \frac{\bar{\nabla}_\perp^2}{2}\right) A(\bar{\boldsymbol{x}}) - \frac{1}{2\pi\bar{\sigma}_x^2}\int_{-\infty}^{0} \tau d\tau \frac{e^{-\bar{\sigma}_\eta^2\tau^2/2 - i\mu\tau}}{\sin^2(\bar{k}_\beta\tau)} \int d^2\bar{x}' A(\bar{\boldsymbol{x}}') \times \exp\left[-\frac{\bar{\boldsymbol{x}}^2 + (\bar{\boldsymbol{x}}')^2 - 2\bar{\boldsymbol{x}}\cdot\bar{\boldsymbol{x}}'\cos(\bar{k}_\beta\tau)}{2\sin^2(\bar{k}_\beta\tau)}\left(i\bar{k}_\beta^2\tau + \frac{1}{\bar{\sigma}_x^2}\right)\right] = 0.$$

$\bar{\nu}$

Detuning / ρ

No Panic!

IT LOOKS UGLY BUT…
The dispersion relation can be expressed as a function of 4 dimensionless parameters

$$\left(\mu - \bar{\nu} + \frac{\bar{\nabla}_\perp^2}{2}\right) A(\bar{\boldsymbol{x}}) - \frac{1}{2\pi\bar{\sigma}_x^2} \int_{-\infty}^{0} \tau d\tau \frac{e^{-\bar{\sigma}_\eta^2 \tau^2/2 - i\mu\tau}}{\sin^2(\bar{k}_\beta \tau)} \int d^2\bar{x}' A(\bar{\boldsymbol{x}}')$$
$$\times \exp\left[-\frac{\bar{\boldsymbol{x}}^2 + (\bar{\boldsymbol{x}}')^2 - 2\bar{\boldsymbol{x}} \cdot \bar{\boldsymbol{x}}' \cos(\bar{k}_\beta \tau)}{2\sin^2(\bar{k}_\beta \tau)} \left(i\bar{k}_\beta^2 \tau + \frac{1}{\bar{\sigma}_x^2}\right)\right] = 0.$$

$$\bar{\sigma}_x = \sigma_x \sqrt{2k_1 k_u \rho} = \frac{2}{\sqrt{3}} \frac{L_R}{L_{G0}}$$

Diffraction parameter

Diffraction negligible if

$$Z_R > L_{G0} \quad \text{or} \quad \bar{\sigma}_x > 1$$

No Panic!

SLAC

IT LOOKS UGLY BUT…
The dispersion relation can be expressed as a function of 4 dimensionless parameters

$$\left(\mu - \bar{\nu} + \frac{\bar{\boldsymbol{\nabla}}_\perp^2}{2}\right) A(\bar{\boldsymbol{x}}) - \frac{1}{2\pi\bar{\sigma}_x^2}\int_{-\infty}^{0} \tau d\tau \frac{e^{-\bar{\sigma}_\eta^2\tau^2/2 - i\mu\tau}}{\sin^2(\bar{k}_\beta\tau)} \int d^2\bar{x}' A(\bar{\boldsymbol{x}}')$$

$$\times \exp\left[-\frac{\bar{\boldsymbol{x}}^2 + (\bar{\boldsymbol{x}}')^2 - 2\bar{\boldsymbol{x}}\cdot\bar{\boldsymbol{x}}'\cos(\bar{k}_\beta\tau)}{2\sin^2(\bar{k}_\beta\tau)}\left(i\bar{k}_\beta^2\tau + \frac{1}{\bar{\sigma}_x^2}\right)\right] = 0.$$

$$\bar{\sigma}_\eta = \frac{\Delta\gamma}{\gamma\rho}$$

Ene ... ameter
(same as 1-D theory!)

No Panic!

IT LOOKS UGLY BUT…
The dispersion relation can be expressed as a function of 4 dimensionless parameters

$$\left(\mu-\bar{\nu}+\frac{\bar{\boldsymbol{\nabla}}_\perp^2}{2}\right)A(\bar{\boldsymbol{x}})-\frac{1}{2\pi\bar{\sigma}_x^2}\int_{-\infty}^{0}\tau d\tau\frac{e^{-\bar{\sigma}_\eta^2\tau^2/2-i\mu\tau}}{\sin^2(\bar{k}_\beta\tau)}\int d^2\bar{x}'A(\bar{\boldsymbol{x}}')$$
$$\times\exp\left[-\frac{\bar{\boldsymbol{x}}^2+(\bar{\boldsymbol{x}}')^2-2\bar{\boldsymbol{x}}\cdot\bar{\boldsymbol{x}}'\cos(\bar{k}_\beta\tau)}{2\sin^2(\bar{k}_\beta\tau)}\left(i\bar{k}_\beta^2\tau+\frac{1}{\bar{\sigma}_x^2}\right)\right]=0.$$

$$\bar{\sigma}_x^2\bar{k}_\beta=\frac{\varepsilon}{2\varepsilon_r}$$

Emittance parameter

$$\varepsilon_r=\lambda_1/(4\pi)$$

Ming Xie Fitting Formula

SLAC

$$L_G = L_{G0}\frac{\sqrt{3}/2}{\mathrm{Im}(\mu_{00})} = L_{G0}(1+\Lambda)$$

Exact and Variational Solutions of
3D Eigenmodes in High Gain FELs

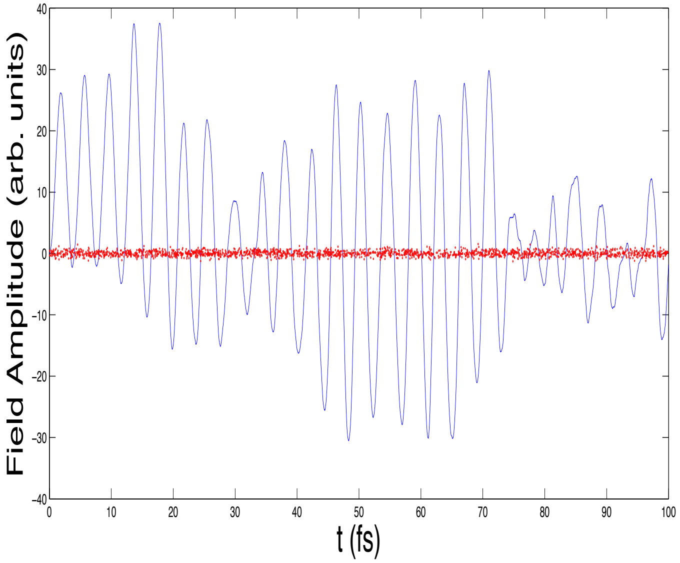
Ming Xie

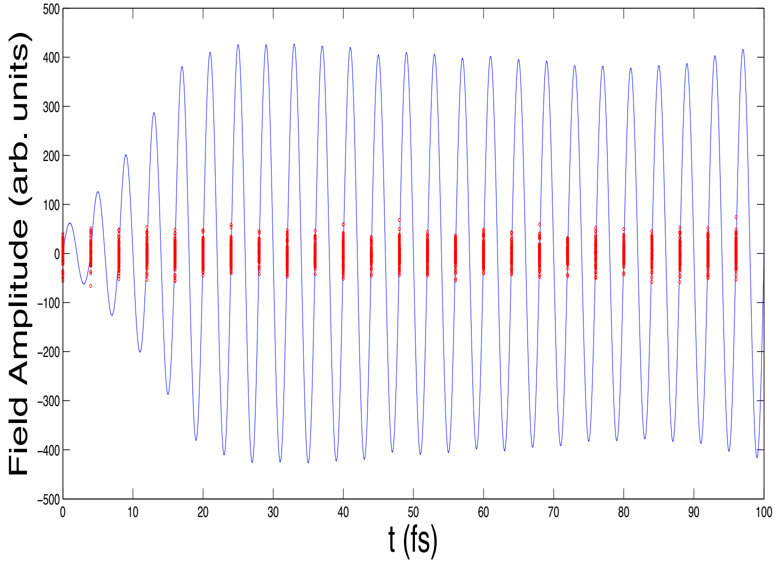
Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA

$$\begin{aligned}\Lambda = & a_1\eta_d^{a_2} + a_3\eta_\varepsilon^{a_4} + a_5\eta_\gamma^{a_6} + a_7\eta_\varepsilon^{a_8}\eta_\gamma^{a_9} \\ & + a_{10}\eta_d^{a_{11}}\eta_\gamma^{a_{12}} + a_{13}\eta_d^{a_{14}}\eta_\varepsilon^{a_{15}} + a_{16}\eta_d^{a_{17}}\eta_\varepsilon^{a_{18}}\eta_\gamma^{a_{19}}\end{aligned}$$

$$\begin{aligned}a_1 &= 0.45, \quad a_2 = 0.57, \quad a_3 = 0.55, \quad a_4 = 1.6, \quad a_5 = 3, \\ a_6 &= 2, \quad a_7 = 0.35, \quad a_8 = 2.9, \quad a_9 = 2.4, \quad a_{10} = 51, \\ a_{11} &= 0.95, \quad a_{12} = 3, \quad a_{13} = 5.4, \quad a_{14} = 0.7, \quad a_{15} = 1.9, \\ a_{16} &= 1140, \quad a_{17} = 2.2 \quad a_{18} = 2.9, \quad a_{19} = 3.2.\end{aligned}$$

Things I Want You To Remember 5 Months From Now



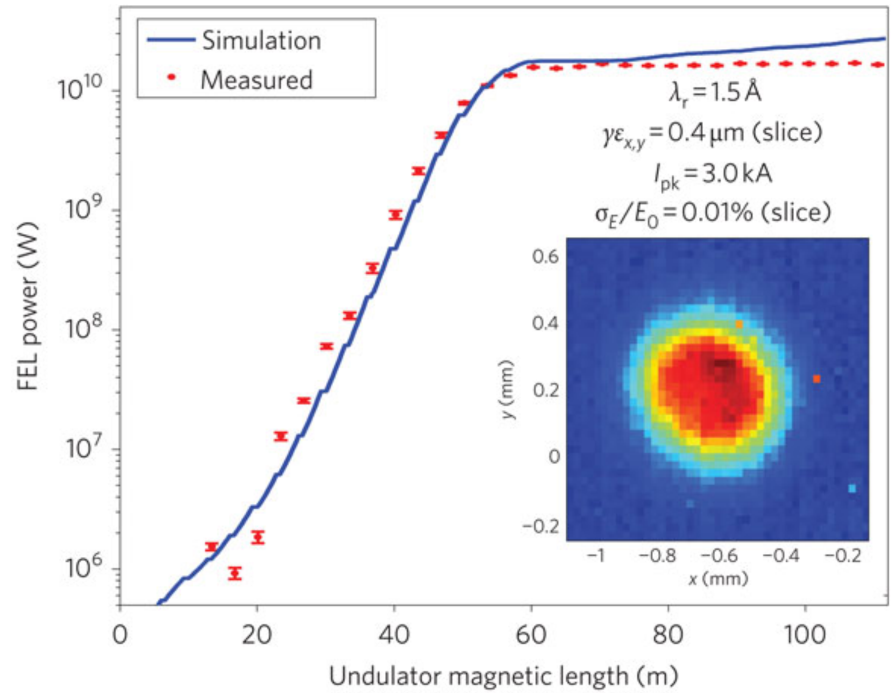


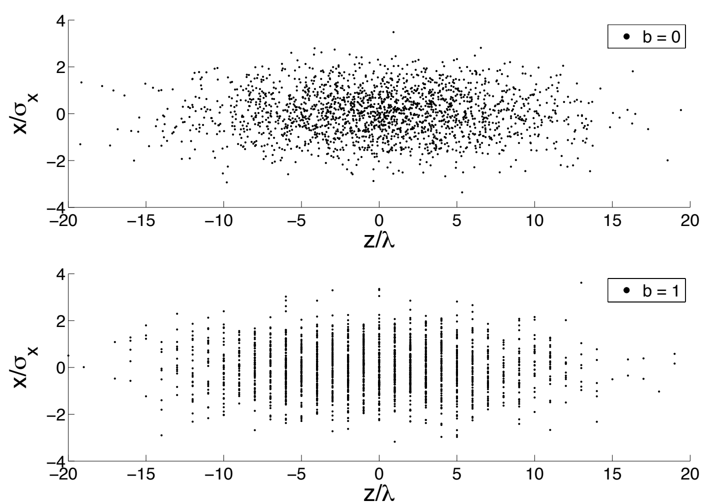
Spontaneous
Power ~ N

Coherent
Power ~ N^2

Things I Want You To Remember 5 Months From Now

FEL goes from spontaneous to coherent emission by means of a collective instability





Things I Want You To Remember 5 Months From Now

$$\rho = (Kk_p/4k_w)^{(2/3)}$$

Rho defines:

-the gain-length of the FEL

-the relative bandwidth of the FEL

-the extraction efficiency of the FEL

-the energy acceptance of the FEL